

# THE HILBERT STACK

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**ABSTRACT.** We show that for *any* locally finitely presented morphism of algebraic stacks  $X \rightarrow S$  with quasi-compact and separated diagonal, there is an algebraic stack  $\underline{\mathrm{HS}}_{X/S}$ , the Hilbert stack, parameterizing proper algebraic stacks with finite diagonal mapping quasi-finitely to  $X$ . The technical heart of this is a generalization of formal GAGA to a non-separated morphism of algebraic stacks, something that was previously unknown for a morphism of schemes. We also employ derived algebraic geometry, in an essential way, to prove the algebraicity of the stack  $\underline{\mathrm{HS}}_{X/S}$ . The Hilbert stack, for a morphism of algebraic spaces, was claimed to exist in [Art74, Appendix §1], but was left unproved due to a lack of foundational results for non-separated algebraic spaces.

## 1. INTRODUCTION

Fix a scheme  $S$  and a morphism of algebraic stacks  $\pi : X \rightarrow S$ . Define the **Hilbert stack**,  $\underline{\mathrm{HS}}_{X/S}$ , to be the  $S$ -stack which sends an  $S$ -scheme  $T$  to the groupoid of quasi-finite and representable morphisms  $(Z \xrightarrow{s} X \times_S T)$ , such that the composition  $Z \xrightarrow{s} X \times_S T \xrightarrow{\pi_T} T$  is proper, flat, and finitely presented with finite diagonal. There is a substack  $\underline{\mathrm{HS}}_{X/S}^{\mathrm{mono}} \subset \underline{\mathrm{HS}}_{X/S}$ , described by monomorphisms  $(Z \xrightarrow{s} X \times_S T)$ . The main results of this paper are

**Theorem 1.** *Fix a scheme  $S$  and a non-separated morphism of noetherian algebraic stacks  $\pi : X \rightarrow S$ , then  $\underline{\mathrm{HS}}_{X/S}^{\mathrm{mono}}$  is never an algebraic stack.*

**Theorem 2.** *Fix a scheme  $S$  and a locally finitely presented morphism of algebraic stacks  $\pi : X \rightarrow S$  with quasi-compact and separated diagonal. Then, the  $S$ -stack  $\underline{\mathrm{HS}}_{X/S}$  is algebraic, locally finitely presented over  $S$ , with quasi-compact and separated diagonal.*

**Theorem 3.** *Fix a scheme  $S$ , a locally finitely presented morphism of algebraic stacks  $X \rightarrow S$  with quasi-finite and separated diagonal; and a proper, flat and finitely presented morphism of algebraic stacks  $Z \rightarrow S$  with finite diagonal. Then, the  $S$ -stack  $T \mapsto \mathrm{HOM}_T(Z \times_S T, X \times_S T)$  is algebraic, locally finitely presented over  $S$ , with quasi-compact and separated diagonal.*

Fix an algebraic stack  $U$ , and a morphism of algebraic stacks  $Y \rightarrow U$ . For another morphism of algebraic stacks  $p : U \rightarrow V$ , define the fibered category  $p_*Y$ , the **restriction of scalars of  $Y$  along  $p$** , by  $(p_*Y)(T) = Y(T \times_V U)$ .

**Theorem 4.** *Fix an algebraic stack  $Z$ , and a locally finitely presented morphism of algebraic stacks  $Y \rightarrow Z$  with quasi-finite and separated diagonal. For any proper, flat, finitely presented morphism of algebraic stacks  $p : Z \rightarrow W$  with finite diagonal, the restriction of scalars of  $Y$  along  $p$ ,  $p_*Y$ , is an algebraic stack, locally finitely presented over  $W$ , with quasi-compact and separated diagonal.*

Theorem 1 is similar to the main conclusion of [LS08], and is included for completeness. In the case that the morphism  $\pi : X \rightarrow S$  is *separated*, the Hilbert stack,  $\underline{\mathrm{HS}}_{X/S}$ , is

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equivalent to the stack of properly supported algebras on  $X$ , which was shown to be algebraic in [Lie06]. Thus the new content of this paper is in the removal of separatedness assumptions from similar theorems in the existing literature. The statement of Theorem 2 for algebraic spaces appeared in [Art74, Appendix §1], but was left unproved due to a lack of foundational results. It is important to note that Theorems 2, 3, and 4 are completely new, even for schemes and algebraic spaces.

Theorems 3 and 4 generalize [Ols06, Thm. 1.1 & 1.5] and [Aok06a] to the non-separated setting, and follow easily from the statement of Theorem 2. Proving Theorem 2 requires an understanding of the infinitesimal deformation theory of the objects in the Hilbert stack, as well as knowledge of how these infinitesimal deformations can be extended to actual deformations.

The obstruction to extending infinitesimal deformations of the Hilbert stack is the dearth of “formal GAGA” type results (cf. [EGA, III, §5]) for *non-separated* schemes, algebraic spaces, and algebraic stacks. In this paper, we will prove a generalization of formal GAGA to non-separated morphisms of algebraic stacks. The proof of our version of non-separated formal GAGA requires the development of a number of foundational results on non-separated spaces, and consumes the majority of the paper.

In the case of a separated morphism of algebraic stacks  $\pi : X \rightarrow S$ , the infinitesimal deformation theory of objects in the stack  $\underline{\mathrm{HS}}_{X/S}$  is closely related to the infinitesimal deformation theory of coherent modules on  $X$ , and is described in [Lie06]. This relation affords computations, as it provides an abelian category, and thus short exact sequences. For a non-separated morphism of algebraic stacks  $\pi : X \rightarrow S$ , we were unable to do this. Thus, the deformation theory for the objects of the Hilbert stack becomes very similar to the deformations of a morphism of *stacks* with a fixed target. The techniques of the cotangent complex of [Ill71] fail to capture this deformation theory. Moreover, in the case that  $X$  is an algebraic stack, accurately describing the deformation theory requires one to work with 2-stacks.

Recasting the Hilbert stack as a moduli problem in derived algebraic geometry, as it appears in [Lur04], allows us to elucidate the necessary deformation-theoretic information. The derived interpretation of the Hilbert stack also permits the application of Lurie’s Representability Theorem [Lur04, Thm. 7.1.6] to prove Theorem 2 in a manner more straightforward than what Artin’s Criterion [Art74, Thm. 5.3] would allow in the generality of a morphism of algebraic stacks.

It is worthwhile to observe that one may construct an algebraic stack parameterizing proper, flat, and finitely presented *algebraic spaces* mapping quasi-finitely to an algebraic stack  $X$  using the non-separated formal GAGA results of this paper, as well as a variant of Artin’s Criterion [Art74, Thm. 5.3] appearing in [Sta06, Prop. 1.1 and 1.2]—this implies Theorems 2, 3, and 4 in the case of algebraic spaces and schemes.

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**1.1. Background.** The most fundamental moduli problem in algebraic geometry is the Hilbert moduli problem for  $\mathbb{P}^N$ : find a scheme which parameterizes flat families of closed subschemes of  $\mathbb{P}^N$ . It was proven by Grothendieck, in [FGA], that this moduli problem has a solution which is a disjoint union of projective schemes.

In general, given a morphism of schemes  $X \rightarrow S$ , one may consider the **Hilbert moduli problem**: find a scheme  $\text{Hilb}_{X/S}$  parameterizing flat families of *closed* subschemes of  $X$ . It is more precisely described by its functor of points: for any scheme  $T$ , a map of schemes  $T \rightarrow \text{Hilb}_{X/S}$  is equivalent to a diagram

$$\begin{array}{ccc} Z & \xrightarrow{\quad} & X \times_S T, \\ & \searrow & \downarrow \\ & & T \end{array}$$

where the morphism  $Z \rightarrow X \times_S T$  is a closed immersion, and the composition  $Z \rightarrow T$  is proper, flat, and finitely presented. Grothendieck, using projective methods, constructed the scheme  $\text{Hilb}_{\mathbb{P}^N/\mathbb{Z}}$ .

In [Art69], M. Artin developed a new approach to constructing moduli spaces. It was proved, by M. Artin in [Art69, Cor. 6.2] and [Art74, Appendix], that the functor  $\text{Hilb}_{X/S}$  had the structure of an algebraic space for any *separated* and locally finitely presented morphism of algebraic spaces  $X \rightarrow S$ . The algebraic space  $\text{Hilb}_{X/S}$  is not, in general, a scheme—even if  $X \rightarrow S$  is a proper morphism of smooth complex varieties. In more recent work, Olsson–Starr [OS03] and Olsson [Ols05] showed that the functor  $\text{Hilb}_{X/S}$  is an algebraic space in the case of a separated and locally finitely presented morphism of algebraic stacks  $X \rightarrow S$ .

A separatedness assumption on a scheme is rarely restrictive to an algebraic geometer. Indeed, most schemes algebraic geometers are interested in are quasi-projective or proper. Let us examine some spaces that arise in the theory of moduli.

**Example 1.1** (Picard Schemes). Let  $C \rightarrow \mathbb{A}^1$  be the family of curves corresponding to a conic degenerating to a node. Consider the Picard scheme  $\text{Pic}_{C/\mathbb{A}^1}$ , which parameterizes families of line bundles on  $C/\mathbb{A}^1$  modulo pullbacks from the base, then  $\text{Pic}_{C/\mathbb{A}^1}$  is not separated. This is worked out in detail in [FGI<sup>+</sup>05, Ex. 9.4.14].

**Example 1.2** (Curves). Let  $\mathcal{U}$  be the stack of *all* curves. That is, a morphism  $T \rightarrow \mathcal{U}$  from a scheme  $T$ , is equivalent to a morphism of algebraic spaces  $C \rightarrow T$  which is proper, flat, finitely presented, with one-dimensional fibers. In particular,  $\mathcal{U}$  parameterizes all singular curves, which could be non-reduced and have many irreducible and connected components. In [Smy09, Appendix B], it was shown that  $\mathcal{U}$  is an algebraic stack, locally of finite presentation over  $\mathbb{Z}$ . The stack  $\mathcal{U}$  is interesting, as Hassett [Has03], Schubert

[Sch91], and Smyth [Smy09] have constructed modular compactifications of  $\mathcal{M}_g$ , different from the classical Deligne-Mumford compactification [DM69], that are open substacks of  $\mathcal{U}$ . The algebraic stack  $\mathcal{U}$  is not separated.

Unlike schemes, non-separatedness of interesting moduli spaces is the norm. Indeed, families of interesting geometric objects tend not to have unique limits. This is precisely the reason why compactifying moduli spaces is an active, and very difficult, area of research.

In [LS08], Lundkvist and Skjelnes showed that for a non-separated morphism of noetherian algebraic spaces  $X \rightarrow S$ , the functor  $\underline{\text{Hilb}}_{X/S}$  is *never* an algebraic space. We will provide an illustrative example of this phenomenon.

**Example 1.3.** Consider the simplest non-separated scheme: the line with the doubled origin. Let's write it as  $\mathbb{A}_s^1 \amalg_{\mathbb{A}_{s=t}^1 - (0)} \mathbb{A}_t^1$ . Now, for a  $y$ -line,  $\mathbb{A}_y^1$ , we have a  $\text{Spec } \mathbb{k}[[x]]$ -morphism  $T_y : \text{Spec } \mathbb{k}[[x]] \rightarrow \mathbb{A}_y^1 \times_{\mathbb{k}} \text{Spec } \mathbb{k}[[x]]$  given by

$$1 \otimes x, y \otimes 1 \mapsto x.$$

Thus, we have an induced map over  $\text{Spec } \mathbb{k}[[x]]$ :

$$\text{Spec } \mathbb{k}[[x]] \xrightarrow{T_s} (\mathbb{A}_s^1 \times_{\mathbb{k}} \text{Spec } \mathbb{k}[[x]] \rightarrow (\mathbb{A}_s^1 \amalg_{\mathbb{A}_{s=t}^1 - (0)} \mathbb{A}_t^1) \times_{\mathbb{k}} \text{Spec } \mathbb{k}[[x]].$$

Now, where  $x = 0$ , this becomes the inclusion of one of the two origins, which is a closed immersion. Where  $x \neq 0$ , this becomes a non-closed immersion. To see this, note that a morphism is a closed immersion, if and only if it is a closed immersion locally on the target. So, for  $\text{Spec } \mathbb{k}[[x]] \rightarrow (\mathbb{A}_s^1 \amalg_{\mathbb{A}_{s=t}^1 - (0)} \mathbb{A}_t^1) \times_{\mathbb{k}} \text{Spec } \mathbb{k}[[x]]$  to be a closed immersion, we need  $\text{Spec } \mathbb{k}[[x]]|_{\mathbb{A}_s^1 \times_{\mathbb{k}} \text{Spec } \mathbb{k}[[x]]} \rightarrow \mathbb{A}_s^1 \times_{\mathbb{k}} \text{Spec } \mathbb{k}[[x]]$ , and  $\text{Spec } \mathbb{k}[[x]]|_{\mathbb{A}_t^1 \times_{\mathbb{k}} \text{Spec } \mathbb{k}[[x]]} \rightarrow \mathbb{A}_t^1 \times_{\mathbb{k}} \text{Spec } \mathbb{k}[[x]]$  to both be closed immersions. Now, it is clear the first map is a closed immersion. The second one is not, since the map is  $\text{Spec } \mathbb{k}((x)) \rightarrow \mathbb{A}_t^1 \times_{\mathbb{k}} \text{Spec } \mathbb{k}[[x]]$ , which is not closed.

Note that if  $X \rightarrow S$  is separated, any monomorphism  $Z \rightarrow X \times_S T$  with the morphism  $Z \rightarrow T$  proper is *automatically* a closed immersion. Thus, for a separated morphism  $X \rightarrow S$ , the stack  $\underline{\text{HS}}_{X/S}^{\text{mono}}$  is equivalent to the Hilbert functor  $\underline{\text{Hilb}}_{X/S}$ . In the case that the morphism  $X \rightarrow S$  is non-separated, they are different. Note that in Example 1.3, the deformed object was still a monomorphism, so will not prove Theorem 1 for the line with the doubled-origin. Let us consider another example.

**Example 1.4.** Consider the line with doubled-origin again, and retain the notation and conventions of Example 1.3. Thus, we have an induced map over  $\text{Spec } \mathbb{k}[[x]]$ :

$$\text{Spec } \mathbb{k}[[x]] \amalg \text{Spec } \mathbb{k}[[x]] \xrightarrow{T_s \amalg T_t} (\mathbb{A}_s^1 \amalg \mathbb{A}_t^1) \times_{\mathbb{k}} \text{Spec } \mathbb{k}[[x]] \rightarrow (\mathbb{A}_s^1 \amalg_{\mathbb{A}_{s=t}^1 - (0)} \mathbb{A}_t^1) \times_{\mathbb{k}} \text{Spec } \mathbb{k}[[x]].$$

Now, where  $x = 0$ , this becomes the inclusion of the doubled point, which is a closed immersion. Where  $x \neq 0$ , this becomes non-monomorphic.

Combining this last example, with the proof of [LS08, Thm. 2.6], one readily obtains the proof of Theorem 1. Thus, for non-separated morphisms of schemes, the obstruction to the existence of a Hilbert scheme is that a monomorphism  $Z \hookrightarrow X$  could be deformed to a non-monomorphism. So, one is forced to consider *maps*  $Z \rightarrow X$ . There exist other versions of Hilbert type moduli problems (in the separated case) in the literature, all of which were shown to be algebraic stacks:

- Vistoli's Hilbert stack, described in [Vis91], parameterizes families of finite and unramified morphisms to a separated stack.

- Lieblich, in [Lie06], showed that the stack of properly supported coherent algebras on a separated and locally finitely presented algebraic stack was algebraic.
- The stack of branchvarieties—parameterizes *varieties* mapping finitely to  $\mathbb{P}^N$ . It was shown that this was an algebraic stack with proper components in [AK10].
- Lieblich, in [Lie06], also considered a generalization of the stack of branchvarieties. It was shown that for certain separated stacks (those with projective coarse moduli or those admitting a proper flat cover by a quasi-projective scheme), the stack of brachvarieties has proper components.
- Hønsen, in [Høn04], constructed a proper algebraic space parameterizing Cohen-Macaulay curves with fixed Hilbert polynomial mapping finitely to projective space, and birationally onto its image.
- Rydh, in [Ryd08], constructed the Hilbert stack of *points* for any morphism of algebraic stacks.

It would be optimal to subsume all of these problems. Hence, the natural class of maps to parameterize is quasi-finite maps  $Z \rightarrow X$ . If we can understand quasi-finite maps, then we can understand unramified morphisms (a natural generalization of immersions), and branchvarieties using standard techniques.

**Example 1.5.** Closed immersions, quasi-compact open immersions, quasi-compact unramified morphisms, and finite morphisms are all examples of quasi-finite morphisms. By Zariski’s Main Theorem [EGA, IV, 18.12.13], any quasi-finite and separated map of schemes  $Z \rightarrow X$  factors as  $Z \rightarrow \bar{Z} \rightarrow X$  where  $Z \rightarrow \bar{Z}$  is an open immersion and  $\bar{Z} \rightarrow X$  is finite.

We now define the **Generalized Hilbert moduli problem**: for a morphism of algebraic stacks  $\pi : X \rightarrow S$ , find an algebraic stack  $\underline{\mathrm{HS}}_{X/S}$  such that a map  $T \rightarrow \underline{\mathrm{HS}}_{X/S}$  is equivalent to the data of a quasi-finite map  $Z \rightarrow X \times_S T$ , with the composition  $Z \rightarrow T$  proper, flat, and finitely presented. This is the fibered category which appears in Theorem 2. The main result in this paper, Theorem 2, is that this stack is algebraic.

**Example 1.6.** If  $X \rightarrow S$  is separated, then any quasi-finite and separated map  $Z \rightarrow X$  which has  $Z \rightarrow S$  proper is automatically finite. Hence,  $\underline{\mathrm{HS}}_{X/S}$  is the stack of properly supported coherent algebras on  $X$ . Lieblich, in [Lie06], showed that  $\underline{\mathrm{HS}}_{X/S}$  is an algebraic stack whenever the morphism  $X \rightarrow S$  is locally of finite presentation and separated.

**1.2. Outline.** In §5, we will prove Theorem 2 using Lurie’s Representability Theorem [Lur04, Thm. 7.1.6]. To apply Lurie’s Theorem, like Artin’s Criterion [Art74, Thm. 5.3], it is necessary to know that infinitesimal deformations of objects in the Hilbert stack can be effectivized. Note that effectivity results for moduli problems related to separated objects usually follow from the formal GAGA results of [EGA, III, §5], and the relevant generalizations to algebraic stacks, which appear in [OS03] and [Ols05]. Since we are concerned with non-separated objects, no previously published effectivity results apply.

In §4, we prove a generalization of formal GAGA for non-separated algebraic stacks, which is the main technical result of this paper. That is, fix an  $I$ -adic noetherian ring  $R$ , and set  $S = \mathrm{Spec} R$  and  $S_n = \mathrm{Spec} R/I^{n+1}$ . For a locally of finite type morphism of algebraic stacks  $\pi : X \rightarrow S$ , with quasi-compact and separated diagonal, let the map  $\pi_n : X_n \rightarrow S_n$  denote the pullback of the map  $\pi$  along the closed immersion  $S_n \hookrightarrow S$ . Suppose that for each  $n \geq 0$ , we have compatible quasi-finite  $S_n$ -morphisms  $s_n : Z_n \rightarrow X_n$  such that the composition  $\pi_n \circ s_n : Z_n \rightarrow S_n$  is proper with finite diagonal. We show that there exists a unique, quasi-finite  $S$ -morphism  $s : Z \rightarrow X$ , such that the composition  $\pi \circ s : Z \rightarrow S$  is proper with finite diagonal, and compatible  $X_n$ -isomorphisms  $Z \times_X X_n \rightarrow Z_n$ .

In §§2 and 3, we will develop techniques to prove the effectivity result of §4. To motivate these techniques, it is instructive to explain part of Grothendieck's proof of [EGA, III, 5.1.4]. So, given an  $I$ -adic noetherian ring  $R$ , let  $S = \operatorname{Spec} R$ , and  $S_n = \operatorname{Spec} R/I^{n+1}$ . For a *proper* morphism of schemes  $f : Y \rightarrow S$ , let the morphism  $f_n : Y_n \rightarrow S_n$  denote the pullback of the morphism  $f$  along the closed immersion  $S_n \hookrightarrow S$ . Suppose that for each  $n \geq 0$ , we have a coherent  $Y_n$ -sheaf  $\mathcal{F}_n$  and isomorphisms  $\mathcal{F}_{n+1}|_{Y_n} \cong \mathcal{F}_n$ , then [EGA, III, 5.1.4] states that there is a coherent  $Y$ -sheaf  $\mathcal{F}$  such that  $\mathcal{F}|_{Y_n} \cong \mathcal{F}_n$ . It is better to think of these adic systems of coherent sheaves as coherent sheaves on formal schemes. For a coherent sheaf  $\mathfrak{F}$  on the formal scheme  $\hat{Y}$ , if there exists a coherent sheaf  $\mathcal{F}$  on the scheme  $Y$  and an isomorphism of coherent sheaves  $\hat{\mathcal{F}} \cong \mathfrak{F}$  on the formal scheme  $\hat{Y}$ , then say that  $\mathfrak{F}$  is **effectivizable**. The effectivity problem is thus recast as: any coherent sheaf  $\mathfrak{F}$  on the formal scheme  $\hat{Y}$  is effectivizable. This is proven using the method of *déviissage* on the category of coherent sheaves  $\mathbf{Coh}(Y)$  on the scheme  $Y$ . The proof consists of the following steps:

- (1) given coherent sheaves  $\mathcal{H}, \mathcal{H}'$  on  $Y$ , then the natural map of  $R$ -modules:

$$\operatorname{Ext}_Y^i(\mathcal{H}, \mathcal{H}') \rightarrow \operatorname{Ext}_{\hat{Y}}^i(\hat{\mathcal{H}}, \hat{\mathcal{H}}')$$

is an isomorphism for all  $i \geq 0$ .

- (2) Show that if we have an exact sequence of coherent sheaves on  $\hat{Y}$ :

$$0 \longrightarrow \mathfrak{H}' \longrightarrow \mathfrak{H} \longrightarrow \mathfrak{H}'' \longrightarrow 0$$

and two of  $\mathfrak{H}', \mathfrak{H}'', \mathfrak{H}$  are effectivizable, then the third is. This follows from the exactness of completion and the  $i = 0, 1$  statements of (1).

- (3) Prove the result for all projective morphisms  $Y \rightarrow S$ .  
 (4) Use Chow's Lemma [EGA, II, 5.6.1] to construct a projective  $S$ -morphism  $p : Y' \rightarrow Y$  that is an isomorphism on a dense open subset of  $Y$  and such that  $Y'$  is  $S$ -projective;  
 (5) Use (3) for the projective morphism  $Y' \rightarrow S$  to show that  $\hat{p}^*\mathfrak{F} \cong \hat{\mathcal{G}}$  for some  $Y'$ -coherent  $\mathcal{G}$ ;  
 (6) Use the Theorem on Formal Functions [EGA, III, 4.1.5] to show that  $(p_*\mathcal{G})^\wedge \cong \hat{p}_*\hat{p}^*\mathfrak{F}$ ,  
 (7) Combine (5) and (6) to see that  $\hat{p}_*\hat{p}^*\mathfrak{F}$  is effectivizable and note that we have an adjunction morphism  $\eta : \mathfrak{F} \rightarrow \hat{p}_*\hat{p}^*\mathfrak{F}$ .  
 (8) Form the two exact sequences:

$$0 \longrightarrow \ker \eta \longrightarrow \mathfrak{F} \xrightarrow{\eta} \operatorname{im} \eta \longrightarrow 0$$

$$0 \longrightarrow \operatorname{im} \eta \longrightarrow \hat{p}_*\hat{p}^*\mathfrak{F} \longrightarrow \operatorname{coker} \eta \longrightarrow 0.$$

By noetherian induction on the topological space  $|Y|$ , we may assume that  $\ker \eta$  and  $\operatorname{coker} \eta$  are effectivizable. Since  $\hat{p}_*\hat{p}^*\mathfrak{F}$  is effectivizable, then applying (2) twice, we conclude that  $\mathfrak{F}$  is effectivizable.

The proof of the non-separated effectivity result in §4 will be very similar to the technique outlined above, once the steps are appropriately reinterpreted. For a non-separated morphism of schemes  $X \rightarrow S$ , instead of the abelian category  $\mathbf{Coh}(X)$ , we consider the (non-abelian) category  $\mathbf{QF}(X)$  which consists of quasi-finite and separated morphisms  $Z \rightarrow X$ . In §2, we will reinterpret (1) and (2) in terms of the existence of pushouts of quasi-finite

and separated morphisms along finite morphisms in the category  $\mathbf{QF}(X)$ . The exactness of the completion functor  $\widehat{(\cdot)} : \mathbf{Coh}(Y) \rightarrow \mathbf{Coh}(\widehat{Y})$  is reinterpreted as the preservation of these coequalizers under completion. The analog of projective morphisms  $Y \rightarrow S$  in (3) and (5) are morphisms which factor as  $Y \rightarrow Y' \rightarrow S$ , where  $Y \rightarrow Y'$  is étale, and  $Y' \rightarrow S$  is projective. The analogue of the Chow Lemma used in (4), is a generalization due to Raynaud-Gruson [RG71, Cor. 5.7.13]. In §3, for a proper morphism  $q : Y' \rightarrow Y$ , we construct an adjoint pair  $(q_!, q^*) : \mathbf{QF}(Y') \rightleftarrows \mathbf{QF}(Y)$  which takes the role of the adjoint pair  $(q^*, q_*) : \mathbf{Coh}(Y) \rightleftarrows \mathbf{Coh}(Y')$ . We also show that the adjoint pair can be constructed for locally noetherian formal schemes, and prove an analogue of (6) and (7) in this setting. In §4, we will combine the results of §§2 and 3 to obtain an analogue of (8).

**1.3. Notation.** We introduce some notation here that will be used throughout the paper. For a category  $\mathcal{C}$  and  $X \in \mathbf{Obj} \mathcal{C}$ , we have the **slice** category  $\mathcal{C}/X$ , with objects the morphisms  $V \rightarrow X$  in  $\mathcal{C}$ , and morphisms commuting diagrams over  $X$ , which are called  $X$ -morphisms. If the category  $\mathcal{C}$  has finite limits and  $f : Y \rightarrow X$  is a morphism in  $\mathcal{C}$ , then for  $(V \rightarrow X) \in \mathbf{Obj}(\mathcal{C}/X)$ , define  $V_Y := V \times_X Y$ . Given a morphism  $p : V' \rightarrow V$ , there is an induced morphism  $p_Y : V'_Y \rightarrow V_Y$ . There is an induced functor  $f^* : \mathcal{C}/X \rightarrow \mathcal{C}/Y : (V \rightarrow X) \mapsto (V_Y \rightarrow Y)$ . Given a higher category  $\mathcal{C}'$ , these notions readily generalize.

Given a ringed space  $U := (|U|, \mathcal{O}_U)$ , a sheaf of ideals  $\mathcal{I} \triangleleft \mathcal{O}_U$ , and a morphism of ringed spaces  $g : V \rightarrow U$ , we define the **pulled back ideal**  $\mathcal{I}_V = \text{im}(g^*\mathcal{I} \rightarrow \mathcal{O}_V) \triangleleft \mathcal{O}_V$ .

Fix a scheme  $S$ , then an algebraic  $S$ -space is a sheaf  $F$  on the big étale site of  $S$ ,  $(\mathbf{Sch}/S)_{\text{ét}}$ , such that the diagonal morphism  $\Delta_F : F \rightarrow F \times_S F$  is represented by schemes, and there is a smooth surjection  $U \rightarrow F$  from an  $S$ -scheme  $U$ . An algebraic  $S$ -stack is a stack  $H$  on  $(\mathbf{Sch}/S)_{\text{ét}}$ , such that the diagonal morphism  $\Delta_H : H \rightarrow H \times_S H$  is represented by algebraic  $S$ -spaces and there is a smooth surjection  $U \rightarrow H$  from an algebraic  $S$ -space  $U$ . Note, we make no separation assumptions on our algebraic stacks. We do show, however, that all algebraic stacks figuring in this paper possess quasi-compact and separated diagonals, thus all of the results of [LMB] apply. We denote the  $(2, 1)$ -category of algebraic stacks by  $\mathbf{AlgStk}$ .

## 2. QUASI-FINITE PUSHOUTS

Fix an algebraic stack  $X$ , and let the 2-category of algebraic stacks over the stack  $X$  be denoted by  $\mathbf{AlgStk}/X$ . Define the full 2-subcategory  $\mathbf{RSch}/X \subset \mathbf{AlgStk}/X$  to have those objects  $Y \xrightarrow{s} X$ , where the morphism  $s$  is schematic. We have three observations here:

- (1) given a morphism  $f : (Z \xrightarrow{s} X) \rightarrow (Z' \rightarrow X)$  in  $\mathbf{RSch}/X$ , then the corresponding  $X$ -map  $f : Z \rightarrow Z'$  is schematic;
- (2) since an  $X$ -morphism  $Z \rightarrow Z'$  is the data of a morphism of stacks  $Z \rightarrow Z'$ , together with a 2-morphism  $\alpha : s \Rightarrow s' \circ f$ , then since  $s$  and  $s'$  are representable, it follows that the 1-morphisms in  $\mathbf{RSch}/X$  have trivial automorphisms, thus  $\mathbf{RSch}/X$  is naturally 2-equivalent to a 1-category;
- (3) if the algebraic stack  $X$  is a scheme, then the natural functor  $\mathbf{Sch}/X \rightarrow \mathbf{RSch}/X$  is an equivalence of categories.

**Definition 2.1.** Let  $X$  be an algebraic stack. Define the following full subcategories of  $\mathbf{RSch}/X$ :

- (1)  $\mathbf{RAff}/X$  has objects the affine morphisms to  $X$ ;
- (2)  $\mathbf{QF}(X)$  has objects the quasi-finite and separated maps to  $X$ .

**2.1. Algebraic Stacks.** For the moment, we will be principally concerned with the existence of pushouts in the category  $\mathbf{QF}(X)$ , where  $X$  is a locally noetherian algebraic stack. Pushouts in algebraic geometry are usually subtle, so we restrict our attention to the types that will be useful in §4. We will need to pay attention to some more general types of colimits in the 2-category of algebraic stacks, as the added flexibility will be useful.

*Remark 2.2.* Note that because we have a fully faithful embedding of categories  $\mathbf{QF}(X) \subset \mathbf{RSch}/X$ , then a categorical colimit in  $\mathbf{RSch}/X$ , which lies in  $\mathbf{QF}(X)$ , is automatically a categorical colimit in  $\mathbf{QF}(X)$ . This will happen frequently in this section.

*Remark 2.3.* It is important to observe that a categorical colimit in  $\mathbf{AlgStk}/X$  is, in general, different to a categorical colimit in  $\mathbf{RSch}/X$ . Indeed, let  $X = \mathrm{Spec} \mathbb{k}$ , and let  $G$  be a finite group, then the  $X$ -stack  $BG$  is the colimit in  $\mathbf{AlgStk}/X$  of the diagram  $[G_X \rightrightarrows X]$ . The colimit in the category  $\mathbf{RSch}/X$  is just  $X$ .

The definitions that follow are closely related to those given in [Ryd07, §2].

**Definition 2.4.** Fix a directed system of algebraic stacks  $\{Z_i\}_{i \in I}$  and consider an algebraic stack  $Z$ , together with compatible morphisms  $\phi_i : Z_i \rightarrow Z$  for every  $i \in I$ . Then we say that the data  $(Z, \{\phi_i\}_{i \in I})$  is a

- (1) **Zariski colimit** if the induced map on topological spaces  $\phi : \varinjlim_i |Z_i| \rightarrow |Z|$  is a homeomorphism (equivalently, the map  $\coprod_{i \in I} \phi_i : \coprod_{i \in I} Z_i \rightarrow Z$  is submersive and the map  $\phi$  is a bijection of sets);
- (2) **weak geometric colimit** if it is a Zariski colimit of the directed system  $\{Z_i\}_{i \in I}$ , and the canonical map of sheaves of rings  $\mathcal{O}_Z \rightarrow \varinjlim_i (\phi_i)_* \mathcal{O}_{Z_i}$  is an isomorphism (in the case of schemes, this is equivalent to the scheme  $Z$  being the colimit of the directed system of schemes  $\{Z_i\}_{i \in I}$  in the category of *ringed spaces*);
- (3) **universal Zariski colimit** if for any algebraic  $Z$ -stack  $Y$ , the data  $(Y, \{(\phi_i)_Y\}_{i \in I})$  is a Zariski colimit of the directed system  $\{(Z_i)_Y\}_{i \in I}$ ;
- (4) **geometric colimit** if it is a universal Zariski and weak geometric colimit of the directed system  $\{Z_i\}_{i \in I}$ ;
- (5) **uniform geometric colimit** if for any *flat*, algebraic  $Z$ -stack  $Y$ , the data  $(Y, \{(\phi_i)_Y\}_{i \in I})$  is a geometric colimit of the directed system  $\{(Z_i)_Y\}_{i \in I}$ .

The following criterion will be useful for verifying when a colimit is a universal Zariski colimit.

**Lemma 2.5.** *Suppose that we have a directed system of algebraic stacks  $\{Z_i\}_{i \in I}$ , an algebraic stack  $Z$ , and compatible morphisms  $\phi_i : Z_i \rightarrow Z$  for all  $i \in I$ . If the map  $\coprod_{i \in I} \phi_i : \coprod_{i \in I} Z_i \rightarrow Z$  is surjective and universally submersive and also*

- (1) *for any geometric point  $\mathrm{Spec} K \rightarrow Z$ , the map  $\phi_K : \varinjlim_i |(Z_i)_K| \rightarrow |\mathrm{Spec} K|$  is an injection of sets; or*
- (2) *there is an algebraic stack  $X$  such that  $Z$  and  $Z_i \in \mathbf{QF}(X)$  for all  $i \in I$ , the maps  $\phi_i$  are  $X$ -maps, and for any geometric point  $\mathrm{Spec} L \rightarrow X$ , the map  $\psi_L : \varinjlim_i |(Z_i)_L| \rightarrow |Z_L|$  is an injection of sets,*

*then  $(Z, \{\phi_i\}_{i \in I})$  is a universal Zariski colimit of the directed system  $\{Z_i\}_{i \in I}$ .*

*Proof.* To show (1), we observe that the universal submersiveness hypothesis on the map  $\coprod_{i \in I} \phi_i : \coprod_{i \in I} Z_i \rightarrow Z$  reduces the statement to showing that for any morphism of algebraic stacks  $Y \rightarrow Z$ , the map  $\phi_Y : \varinjlim_i |(Z_i)_Y| \rightarrow |Y|$  is a bijection of sets, which will follow if the map  $\phi_K : \varinjlim_i |(Z_i)_K| \rightarrow |\mathrm{Spec} K|$  is bijective for any geometric point  $\mathrm{Spec} K \rightarrow Y$ .



By assumption, we know that  $\phi_K$  is injective, for the surjectivity, we note that we have a commutative diagram of sets:

$$\begin{array}{ccc} \coprod_i |(Z_i)_K| & \xrightarrow{\alpha_K} & \varinjlim_i |(Z_i)_K|, \\ & \searrow \Pi_i \phi_i & \downarrow \phi_K \\ & & |\mathrm{Spec} K| \end{array}$$

where  $\alpha_K$  and  $\Pi_i \phi_i$  are surjective, thus  $\phi_K$  is also surjective. For (2), given a geometric point  $\mathrm{Spec} L \rightarrow X$ , then  $Z(\mathrm{Spec} L) = |Z_L|$  and  $Z_i(\mathrm{Spec} L) = |(Z_i)_L|$ , thus we may apply the criterion of (1) to obtain the claim.  $\square$

The exactness of flat pullback of sheaves easily proves

**Lemma 2.6.** *Suppose that we have a finite directed system of algebraic stacks  $\{Z_i\}_{i \in I}$ , then a geometric colimit  $(Z, \{\phi_i\}_{i \in I})$  is a uniform geometric colimit.*

Also, note that weak geometric colimits in the category of algebraic stacks are not unique and thus are not categorical colimits—the example in Remark 2.3 demonstrates this. In the setting of schemes, however, we have

**Lemma 2.7.** *Given a directed system of schemes  $\{Z_i\}_{i \in I}$ , with a weak geometric colimit  $(Z, \{\phi_i\}_{i \in I})$ , then it is a colimit in the category of locally ringed spaces, thus is a colimit in the category of schemes.*

To obtain a similarly useful result for stacks, we will need some relative notions, unlike the previous definitions which were all absolute.

**Definition 2.8.** Fix an algebraic stack  $X$  and let  $\{Z_i\}_{i \in I}$  be a directed system of objects in  $\mathbf{RSch}/X$ . Suppose that  $Z \in \mathbf{RSch}/X$ , and we have compatible  $X$ -morphisms  $\phi_i : Z_i \rightarrow Z$  for all  $i \in I$ . We say that the data  $(Z, \{\phi_i\}_{i \in I})$  is a **uniform categorical colimit in  $\mathbf{RSch}/X$**  if for any *flat*, algebraic  $X$ -stack  $Y$ , the data  $(Z_Y, \{(\phi_i)_Y\}_{i \in I})$  is the categorical colimit of the directed system  $\{(Z_i)_Y\}_{i \in I}$  in  $\mathbf{RSch}/Y$ .

Combining Lemmata 2.6 and 2.7 we obtain

**Corollary 2.9.** *Let  $X$  be an algebraic stack, and suppose that  $\{Z_i\}_{i \in I}$  is a finite directed system of objects in  $\mathbf{RSch}/X$ , then a geometric colimit  $(Z, \{\phi_i\}_{i \in I})$  in  $\mathbf{RSch}/X$  is also a uniform geometric and uniform categorical colimit in  $\mathbf{RSch}/X$ .*

*Proof.* By Lemma 2.6, it suffices to show that  $Z$  is a categorical colimit. Let  $U \rightarrow X$  be a smooth surjection from a scheme. By Lemmata 2.6 and 2.7, we see that  $(Z_U, \{(\phi_i)_U\}_{i \in I})$  is a categorical colimit of the system  $\{(Z_i)_U\}_{i \in I}$  in  $\mathbf{RSch}/U$ , and remains so after flat schematic base change on  $U$ . Suppose that we have compatible  $X$ -morphisms  $\alpha_i : Z_i \rightarrow W$ , then we want to show that there is a unique  $X$ -morphism  $\alpha : Z \rightarrow W$  which is compatible with this data. Let  $R = U \times_X U$ , and let  $s, t : R \rightarrow U$  denote the two projections, then since  $Z_U$  is the categorical colimit in  $\mathbf{RSch}/U$  there is a unique  $X$ -morphism  $\beta_U : Z_U \rightarrow W_U$  which is compatible with  $(\alpha_i)_U : (Z_i)_U \rightarrow W_U$ .

Suppose for the moment that  $X$  is an algebraic space, then  $R$  is a scheme and the maps  $s, t : R \rightarrow U$  are morphisms of schemes. In particular, we know that  $Z_R$  is a categorical colimit in  $\mathbf{RSch}/R$  and so there is a unique morphism  $\beta_R : Z_R \rightarrow W_R$  which is compatible with the morphisms  $(\alpha_i)_R : (Z_i)_R \rightarrow W_R$ . Noting that  $(\beta_U) \times_{U, s} R$  and  $(\beta_U) \times_{U, t} R$  are also such maps, we conclude that these maps are actually equal (as maps of algebraic spaces) and so by smooth descent, we conclude that there is a unique  $X$ -morphism  $\alpha : Z \rightarrow W$ ,

and so  $(Z, \{\phi_i\}_{i \in I})$  is a categorical colimit if  $X$  is an algebraic space. Applying Lemma 2.6, one is able to conclude that  $(Z, \{\phi_i\}_{i \in I})$  remains a categorical colimit after flat *representable* base change on  $X$ , for all algebraic spaces  $X$ .

Now, returning to the case that  $X$  is an algebraic stack, we know that  $s, t : R \rightarrow U$  are flat representable morphisms, repeating the descent argument given above shows that  $(Z, \{\phi_i\}_{i \in I})$  is a categorical colimit in  $\mathbf{RSch}/X$ .  $\square$

The following result is a generalization of [Kol08, Lem. 17], which treats the case of a finite equivalence relation of schemes.

**Theorem 2.10.** *Let  $X$  be a locally noetherian algebraic stack, and suppose that  $[Z' \xleftarrow{t'} Z \xrightarrow{t''} Z'']$  is a diagram in  $\mathbf{RSch}/X$ . If  $t'$  and  $t''$  are finite, and  $Z, Z', Z'' \in \mathbf{QF}(X)$ , this diagram has a uniform categorical colimit  $Z'''$  in  $\mathbf{RSch}/X$ , which is a uniform geometric colimit and furthermore,  $Z''' \in \mathbf{QF}(X)$ . If  $t'$  (or  $t''$ ) is a closed immersion, the cocartesian diagram in  $\mathbf{RSch}/X$ :*

$$\begin{array}{ccc} Z & \xrightarrow{t'} & Z' \\ t'' \downarrow & & \downarrow \\ Z'' & \longrightarrow & Z''' \end{array}$$

*is cartesian.*

We will need some lemmata to prove Theorem 2.10.

**Lemma 2.11.** *Let  $X$  be an algebraic stack, and suppose that  $\{Z_i\}_{i \in I}$  is a finite directed system in  $\mathbf{RAff}/X$ , then this system has a categorical colimit in  $\mathbf{RAff}/X$ , whose formation commutes with flat base change on  $X$ . If the system is of the form  $[Z' \xleftarrow{t'} Z \xrightarrow{t''} Z'']$  and  $t'$  (or  $t''$ ) is a closed immersion, then the induced cocartesian square in  $\mathbf{RAff}/X$  is also cartesian.*

*Proof.* By [LMB, Prop. 14.2.4], there is an anti-equivalence of categories between  $\mathbf{RAff}/X$  and the category of quasicoherent sheaves of  $\mathcal{O}_X$ -algebras, which commutes with arbitrary change of base, and is given by  $(Z \xrightarrow{s} X) \mapsto (\mathcal{O}_X \xrightarrow{s^\#} s_* \mathcal{O}_Z)$ . Since the category of quasicoherent  $\mathcal{O}_X$ -algebras has finite limits, it follows that if  $s_i : Z_i \rightarrow X$  denotes the structure map of  $Z_i$ , then the categorical colimit is  $Z = \mathrm{Spec}_X \varinjlim_i (s_i)_* \mathcal{O}_{Z_i}$ . Since flat pullback of sheaves is exact, the formation of this colimit clearly commutes with flat base change on  $X$ . If  $t'$  (or  $t''$ ) is a closed immersion, the isomorphism of rings  $A \otimes_{A \times_{B/J} B} B \cong B/J$  for ring maps  $[A \rightarrow B/J \leftarrow B]$  shows that the pushout diagram is also cartesian.  $\square$

**Lemma 2.12.** *Let  $X$  be a locally noetherian algebraic stack, and suppose that  $[Z' \xleftarrow{t'} Z \xrightarrow{t''} Z'']$  is a diagram in  $\mathbf{RSch}/X$ , where  $Z, Z', Z''$  are all finite over  $X$ , then this diagram has a uniform categorical and uniform geometric colimit  $Z'''$  in  $\mathbf{RSch}/X$ , which is finite over  $X$ . If  $t'$  (or  $t''$ ) is a closed immersion, then the induced cocartesian square in  $\mathbf{RSch}/X$  is also cartesian.*

*Proof.* By Lemma 2.11, the diagram has a categorical colimit  $Z'''$  in  $\mathbf{RAff}/X$ , and since  $X$  is locally noetherian, one readily deduces that  $Z'''$  is finite over  $X$ . By Corollary 2.9, it remains to show that  $Z'''$  is a geometric colimit. Note that since  $Z' \amalg Z'' \rightarrow Z'''$  is dominant and finite, it is surjective, universally closed and thus universally submersive. First, we show that  $Z'''$  is a universal Zariski colimit using the criterion of Lemma 2.5(2)).

We now follow [Kol08, Lem. 17]. Let  $\bar{x} : \mathrm{Spec} \mathbb{k} \rightarrow X$  be a geometric point, then by [EGA, 0<sub>III</sub>, 10.3.1] this map factors as  $\mathrm{Spec} \mathbb{k} \xrightarrow{\bar{x}^1} X^1 \xrightarrow{p} X$ , where  $p$  is flat and  $X^1$  is the spectrum of a maximal-adically complete, local noetherian ring with residue field  $\mathbb{k}$ . Since

the algebraic stack  $Z'''$  is a uniform categorical colimit in  $\mathbf{RAff}/X$ , we may replace  $X$  by  $X^1$ , and we denote the unique closed point of  $X$  by  $x$ . For a finite  $X$ -scheme  $U$ , let  $\pi_0(U)$  be its set of connected components. The assumptions on  $X$  guarantee that there is a unique, universal homeomorphism  $h_U : U_x \rightarrow \coprod_{m \in \pi_0(U)} \{x\}$  which is functorial with respect to  $U$ . In particular, there is a unique factorization  $U \xrightarrow{s_U} \coprod_{m \in \pi_0(U)} X \rightarrow X$  such that  $(s_U)_x = h_U$ . Since  $\pi_0(-)$  is a functor, we obtain a diagram:

$$\left[ \coprod_{m \in \pi_0(Z')} X \xleftarrow{\pi_0(t')} \coprod_{m \in \pi_0(Z)} X \xrightarrow{\pi_0(t'')} \coprod_{m \in \pi_0(Z'')} X \right]$$

in  $\mathbf{RAff}/X$ , and we let  $X'''$  be the categorical colimit of this diagram in  $\mathbf{RAff}/X$ . There is thus a canonical map  $\mu : Z''' \rightarrow X'''$ , together with a functorial bijection of sets  $\pi_0(Z') \amalg_{\pi_0(Z)} \pi_0(Z'') \rightarrow \pi_0(X''')$ . In particular, the bijection  $\nu_x : |Z'_x| \amalg_{|Z_x|} |Z''_x| \rightarrow |X'''_x|$  factors as  $|Z'_x| \amalg_{|Z_x|} |Z''_x| \xrightarrow{\psi_x} |Z'''_x| \xrightarrow{\mu_x} |X'''_x|$  and thus  $\psi_x : |Z'_x| \amalg_{|Z_x|} |Z''_x| \rightarrow |Z'''_x|$  is injective. Hence, we have shown that  $Z'''$  is a universal Zariski colimit and it remains to show that  $Z'''$  has the correct functions.

Let  $m' : Z' \rightarrow Z'''$ ,  $m'' : Z'' \rightarrow Z'''$  denote the canonical maps and let  $m : Z \rightarrow Z'''$  denote the induced map from  $Z$ . Now there is a canonical morphism of sheaves of  $\mathcal{O}_X$ -algebras  $\epsilon : \mathcal{O}_{Z'''} \rightarrow m'_* \mathcal{O}_{Z'} \times_{m_* \mathcal{O}_Z} m''_* \mathcal{O}_{Z''}$ , which we have to show is an isomorphism. Let  $h''' : Z''' \rightarrow X$  denote the structure map, then by functoriality, we have an induced morphism of sheaves of  $\mathcal{O}_X$ -algebras:

$$\epsilon_2 : h'''_* \mathcal{O}_{Z'''} \xrightarrow{h'''_* \epsilon} h'''_* (m'_* \mathcal{O}_{Z'} \times_{m_* \mathcal{O}_Z} m''_* \mathcal{O}_{Z''}) \xrightarrow{\epsilon_1} h'''_* m'_* \mathcal{O}_{Z'} \times_{h'''_* m_* \mathcal{O}_Z} h'''_* m''_* \mathcal{O}_{Z''}.$$

Since the functor  $h'''_*$  is left exact,  $\epsilon_1$  is an isomorphism; by construction of  $Z'''$  the map  $\epsilon_2$  is an isomorphism and so  $h'''_* \epsilon$  is an isomorphism. Since  $h'''$  is affine, the functor  $h'''_*$  is faithfully exact and we conclude that the map  $\epsilon$  is an isomorphism of sheaves.  $\square$

*Proof of Theorem 2.10.* By Corollary 2.9, it suffices to construct a geometric colimit. By hypothesis, the morphisms  $s : Z \rightarrow X$ ,  $s' : Z' \rightarrow X$ ,  $s'' : Z'' \rightarrow X$  are quasi-finite, separated, and representable. Using Zariski's Main Theorem [LMB, Thm. 16.5(ii)], there are finite  $X$ -morphisms  $\bar{s}' : W' \rightarrow X$ ,  $\bar{s}'' : W'' \rightarrow X$  and open, dense immersions  $\iota' : Z' \hookrightarrow W'$  and  $\iota'' : Z'' \hookrightarrow W''$ . Let  $W_0 = W' \times_X W''$ , and apply Zariski's Main Theorem again [*loc. cit.*] to  $Z \rightarrow W_0$  and let  $W$  be finite over  $W_0$  with  $\iota : Z \hookrightarrow W$  open and dense. We thus obtain finite maps  $\bar{\iota}' : W \rightarrow W'$ ,  $\bar{\iota}'' : W \rightarrow W''$ . Note that if  $t'$  (resp.  $t''$ ) is a closed immersion, then we can choose  $\bar{\iota}'$  (resp.  $\bar{\iota}''$ ) to be a closed immersion. To fix notation, we let  $\bar{s} : W \rightarrow X$  denote the induced structure map.

We observe by Lemma 2.12, that the diagram  $[W' \xleftarrow{\bar{\iota}'} W \xrightarrow{\bar{\iota}''} W'']$  has a uniform geometric and uniform categorical colimit in  $\mathbf{QF}(X)$ ,  $\bar{s}''' : W''' \rightarrow X$ . We take  $|Z'''|$  to be the set-theoretic image of  $|Z'| \amalg |Z''|$  in  $|W'''|$ . We claim that  $|Z'''|$  is an open subset of  $|W'''|$ . Since  $W' \amalg W'' \rightarrow W'''$  is universally submersive, to check that  $|Z'''|$  is an open subset of  $|W'''|$ , it suffices to show that the preimage of  $|Z'''|$  under  $W' \amalg W'' \rightarrow W'''$  is precisely  $|Z'| \amalg |Z''|$  (which is open). This last claim will follow if we show that the pullback of  $\iota' : Z' \hookrightarrow W'$  and  $\iota'' : Z'' \hookrightarrow W''$  by  $W \rightarrow W'$ ,  $W''$  is  $Z$ . Indeed, this is nothing other than the definition of the colimit of topological spaces. By symmetry, it suffices to prove this claim for  $Z'$ . Now, we have canonical maps  $Z \xrightarrow{\alpha} Z' \times_{W'} W \xrightarrow{\beta} W$  and since the maps  $\beta \circ \alpha$  and  $\beta$  are open immersions,  $\alpha : Z \rightarrow Z' \times_{W'} W$  is an open immersion. We also see that  $Z \xrightarrow{\alpha} Z' \times_{W'} W \rightarrow Z'$  is a composition of finite morphisms thus  $\alpha : Z \rightarrow Z' \times_{W'} W$  is open

and closed. Since  $\beta \circ \alpha : Z \rightarrow W$  is dense, we conclude that  $Z = Z' \times_{W'} W$ . In particular, we have shown that the continuous map  $\psi_X : |Z'| \amalg_{|Z|} |Z''| \rightarrow |Z'''|$  is a homeomorphism.

Hence, we let  $\iota''' : Z''' \hookrightarrow W'''$  be the open immersion associated to the open subset  $|Z'''| \subset |W'''|$  and it remains to show that  $(Z''' \xrightarrow{s'''} X) \in \mathbf{QF}(X)$  is a geometric colimit. We obtain a commutative diagram in  $\mathbf{QF}(X)$ :

$$(1) \quad \begin{array}{ccccc} & & Z & \longrightarrow & Z' \\ & \swarrow & \downarrow & & \swarrow \\ W & \longrightarrow & W' & & \\ \downarrow & & \downarrow & & \downarrow \\ & & Z'' & \longrightarrow & Z''' \\ \swarrow & & \downarrow & & \swarrow \\ W'' & \longrightarrow & W''' & & \end{array}$$

where all sides are cartesian, except for the front and back. In particular, if  $t'$  (or  $t''$ ) is a closed immersion, then the front square is also cartesian (by Lemma 2.12), thus so is the back square. In particular, for every morphism of algebraic stacks  $Y \rightarrow X$ , we have a bijection  $\bar{\psi}_Y : |W'_Y| \amalg_{|W_Y|} |W''_Y| \rightarrow |W'''_Y|$  and a map  $\psi_Y : |Z'_Y| \amalg_{|Z_Y|} |Z''_Y| \rightarrow |Z'''_Y|$ . By Lemma 2.5(2), it suffices to show that  $\psi_Y$  is injective when  $Y = \operatorname{Spec} K$  and  $K$  is an algebraically closed field. Note that we have the commutative diagram of sets:

$$\begin{array}{ccc} |Z'_Y| \amalg_{|Z_Y|} |Z''_Y| & \xrightarrow{\psi_Y} & |Z'''_Y| \\ \delta_Y \downarrow & & \downarrow \iota'''_Y \\ |W'_Y| \amalg_{|W_Y|} |W''_Y| & \xrightarrow{\bar{\psi}_Y} & |W'''_Y| \end{array}$$

and from the injectivity of the map  $\bar{\psi}_Y$ , it suffices to prove that the map  $\delta_Y$  is injective, which is obvious from the injectivity of  $\iota_Y, \iota'_Y, \iota''_Y$ ; thus  $Z'''$  is a universal Zariski colimit. To show that  $Z'''$  has the correct functions, we let  $m : Z \rightarrow Z'''$ ,  $m' : Z' \rightarrow Z'''$ ,  $m'' : Z'' \rightarrow Z'''$ ,  $\bar{m} : W \rightarrow W'''$ ,  $\bar{m}' : W' \rightarrow W'''$  and  $\bar{m}'' : W'' \rightarrow W'''$  denote the canonical maps. We observe that there are canonical isomorphisms:

$$\mathcal{O}_{Z'''} \cong \iota'''^{-1} \mathcal{O}_{W'''} \cong \iota'''^{-1} (\bar{m}'_* \mathcal{O}_{W'}) \times_{\iota'''^{-1} (\bar{m}_* \mathcal{O}_W)} \iota'''^{-1} (\bar{m}''_* \mathcal{O}_{W''}).$$

Hence, it suffices to show that  $\iota'''^{-1} \bar{m}'_* \mathcal{O}_{W'} = m'_* \mathcal{O}_{Z'}$  (and similarly for the other objects), but this is obvious from the definitions and the cartesian squares in (1).  $\square$

**2.2. Formal Schemes.** Here, we will extend Theorem 2.10 to locally noetherian formal schemes. Denote the category of formal schemes by  $\mathbf{FSch}$ . We require some more definitions that are analogous to those given in §2.1.

**Definition 2.13.** Consider a directed system of formal schemes  $\{\mathfrak{Z}_i\}_{i \in I}$ , a formal scheme  $\mathfrak{Z}$ , and suppose that we have compatible map  $\varphi_i : \mathfrak{Z}_i \rightarrow \mathfrak{Z}$  for every  $i \in I$ . Then we say that the data  $(\mathfrak{Z}, \{\varphi_i\}_{i \in I})$  is a

- (1) **formal Zariski colimit** if the induced map on topological spaces  $\varinjlim_i |\mathfrak{Z}_i| \rightarrow |\mathfrak{Z}|$  is a homeomorphism;
- (2) **formal weak geometric colimit** if it is a formal Zariski colimit and the canonical map of sheaves of rings  $\mathcal{O}_{\mathfrak{Z}} \rightarrow \varinjlim_i (\varphi_i)_* \mathcal{O}_{\mathfrak{Z}_i}$  is a topological isomorphism, where we give the latter sheaf of rings the limit topology (this is nothing other than  $\mathfrak{Z}$  being the colimit in the category of topologically ringed spaces);

- (3) **universal formal Zariski colimit** if for any adic locally of finite type formal  $\mathfrak{Z}$ -scheme  $\mathfrak{Y}$ ,  $(\mathfrak{Y}, \{(\varphi_i)_{\mathfrak{Y}}\}_{i \in I})$  is the formal Zariski colimit of the directed system  $\{(\mathfrak{Z}_i)_{\mathfrak{Y}}\}_{i \in I}$ ;
- (4) **formal geometric colimit** if it is a universal formal Zariski and a formal weak geometric colimit of the directed system  $\{\mathfrak{Z}_i\}_{i \in I}$ ;
- (5) **uniform formal geometric colimit** if for any adic flat and locally of finite type formal  $\mathfrak{Z}$ -scheme  $\mathfrak{Y}$ ,  $(\mathfrak{Y}, \{(\varphi_i)_{\mathfrak{Y}}\}_{i \in I})$  is a formal geometric colimit of the directed system  $\{(\mathfrak{Z}_i)_{\mathfrak{Y}}\}_{i \in I}$ .

We have two lemmata which are formal analogues of Lemmata 2.6 and 2.7.

**Lemma 2.14.** *Given a directed system of formal schemes  $\{\mathfrak{Z}_i\}_{i \in I}$ , then a weak geometric colimit  $(\mathfrak{Z}, \{\varphi_i\}_{i \in I})$  is a colimit in the category of topologically locally ringed spaces, thus is a colimit in the category  $\mathbf{FSch}$ .*

**Lemma 2.15.** *Given a finite directed system of locally noetherian formal schemes  $\{\mathfrak{Z}_i\}_{i \in I}$ , with a formal geometric colimit  $(\mathfrak{Z}, \{\varphi_i\}_{i \in I})$  such that  $\varphi_i$  is finite for all  $i \in I$ , then it is a uniform formal geometric colimit.*

*Proof.* Let  $\varpi : \mathfrak{Y} \rightarrow \mathfrak{Z}$  be an adic flat and locally of finite type morphism of locally noetherian formal schemes, then it remains to show that the canonical map  $\vartheta_{\mathfrak{Y}} : \mathcal{O}_{\mathfrak{Y}} \rightarrow \varprojlim_i [(\varphi_i)_{\mathfrak{Y}}]_* \mathcal{O}_{(\mathfrak{Z}_i)_{\mathfrak{Y}}}$  is a topological isomorphism. Since  $\varphi_i$  is finite for all  $i$ , and we are taking a finite limit, we conclude that it suffices to show that  $\vartheta_{\mathfrak{Y}}$  is an isomorphism of coherent  $\mathcal{O}_{\mathfrak{Y}}$ -modules. By hypothesis, the map  $\vartheta_{\mathfrak{Z}} : \mathcal{O}_{\mathfrak{Z}} \rightarrow \varprojlim_i (\varphi_i)_* \mathcal{O}_{\mathfrak{Z}_i}$  is an isomorphism, and since  $\varpi$  is adic flat,  $\varpi^*$  is an exact functor from coherent  $\mathcal{O}_{\mathfrak{Z}}$ -modules to coherent  $\mathcal{O}_{\mathfrak{Y}}$ -modules, thus commutes with finite limits. Hence, we see that  $\vartheta_{\mathfrak{Y}}$  factors as the following sequence of isomorphisms:

$$\mathcal{O}_{\mathfrak{Y}} \cong \varpi^* \mathcal{O}_{\mathfrak{Z}} \cong \varprojlim_i \varpi^* [(\varphi_i)_* \mathcal{O}_{\mathfrak{Z}_i}] \cong \varprojlim_i [(\varphi_i)_{\mathfrak{Y}}]_* \mathcal{O}_{(\mathfrak{Z}_i)_{\mathfrak{Y}}}. \quad \square$$

**Definition 2.16.** Fix a formal scheme  $\mathfrak{X}$  and let  $\{\mathfrak{Z}_i\}_{i \in I}$  be a directed system of objects in  $\mathbf{FSch}/\mathfrak{X}$ . Let  $\mathfrak{Z} \in \mathbf{FSch}/\mathfrak{X}$  and suppose we have compatible  $\mathfrak{X}$ -morphisms  $\varphi_i : \mathfrak{Z}_i \rightarrow \mathfrak{Z}$ . Then we say that  $(\mathfrak{Z}, \{\varphi_i\}_{i \in I})$  is a **uniform categorical colimit in  $\mathbf{FSch}/\mathfrak{X}$**  if for any adic flat and locally of finite type formal  $\mathfrak{X}$ -scheme  $\mathfrak{Y}$ ,  $(\mathfrak{Z}_{\mathfrak{Y}}, \{(\varphi_i)_{\mathfrak{Y}}\}_{i \in I})$  is the categorical colimit of the directed system  $\{(\mathfrak{Z}_i)_{\mathfrak{Y}}\}_{i \in I}$  in  $\mathbf{FSch}/\mathfrak{Y}$ .

**Definition 2.17.** Let  $\mathfrak{X}$  be a formal scheme. Define  $\mathbf{QF}(\mathfrak{X})$  to be the category whose objects are adic, quasi-finite, and separated maps  $(\mathfrak{Z} \xrightarrow{\sigma} \mathfrak{X})$ . A morphism in  $\mathbf{QF}(\mathfrak{X})$  is an  $\mathfrak{X}$ -morphism  $f : (\mathfrak{Z} \xrightarrow{\sigma} \mathfrak{X}) \rightarrow (\mathfrak{Z}' \xrightarrow{\sigma'} \mathfrak{X})$ .

For a scheme  $X$ , and a closed subset  $|V| \subset |X|$ , we define the **completion functor**

$$c_{X, |V|} : \mathbf{Sch}/X \rightarrow \mathbf{FSch}/\widehat{X}_{/|V|} : (Z \xrightarrow{s} X) \mapsto (\widehat{Z}_{/|s|^{-1}|V|} \rightarrow \widehat{X}_{/|V|}).$$

Note that restricting  $c_{X, |V|}$  to  $\mathbf{QF}(X)$  has essential image contained in  $\mathbf{QF}(\widehat{X}_{/|V|})$ .

**Theorem 2.18.** *Let  $X$  be a locally noetherian scheme, suppose that  $|V| \subset |X|$  is a closed subset, and let  $[Z' \xleftarrow{t'} Z \xrightarrow{t''} Z'']$  be a diagram in  $\mathbf{QF}(X)$  with  $t', t''$  finite. If the scheme  $Z'''$  denotes the categorical colimit of the diagram in  $\mathbf{Sch}/X$ , which exists by Theorem 2.10, then  $c_{X, |V|}(Z''')$  is a uniform categorical and uniform formal geometric colimit of the diagram*

$$[c_{X, |V|}(Z') \xleftarrow{\widehat{t}'} c_{X, |V|}(Z) \xrightarrow{\widehat{t}''} c_{X, |V|}(Z'')]$$

in  $\mathbf{FSch}/\widehat{X}_{/|V|}$ .

For a locally noetherian scheme  $X$ , and a closed subset  $|V| \subset |X|$ , if a formal  $\widehat{X}_{/|V|}$ -scheme  $\mathfrak{Z}$  belongs to the essential image of the functor  $c_{X,|V|}$ , we will say that  $\mathfrak{Z}$  is **effectivizable**. Theorem 2.18 trivially implies

**Corollary 2.19.** *If  $X$  is a locally noetherian scheme, and  $\mathfrak{Z} \in \mathbf{QF}(\widehat{X}_{/|V|})$  is obtained by pushing out effectivizable objects of  $\mathbf{QF}(\widehat{X}_{/|V|})$  along finite morphisms, then  $\mathfrak{Z}$  is effectivizable.*

*Proof of Theorem 2.18.* By Lemmata 2.14 and 2.15, it suffices to show that  $c_{X,|V|}(Z''')$  is a formal geometric colimit of the diagram  $[c_{X,|V|}(Z') \leftarrow c_{X,|V|}(Z) \rightarrow c_{X,|V|}(Z'')]$ . We first show that  $c_{X,|V|}(Z''')$  is a universal formal Zariski colimit. Let  $\omega : \mathfrak{Y} \rightarrow \mathfrak{X}$  be an adic locally of finite type morphism of locally noetherian formal schemes. Take  $\mathcal{I}$  denote a coherent sheaf of radical ideals defining  $|V| \subset |X|$ , then  $(\omega^{-1}\mathcal{I})_{\mathfrak{O}_{\mathfrak{Y}}}$  is an ideal of definition of  $\mathfrak{Y}$ . Let  $X_n = V(\mathcal{I}^n) \subset X$ ,  $Y_n = \mathfrak{Y} \times_X X_n$ ,  $Z_n = Z \times_X X_n$ ,  $Z'_n = Z' \times_X X_n$ ,  $Z''_n = Z'' \times_X X_n$ , then by Theorem 2.10 the map of topological spaces  $|Z'_n|_{Y_0} \coprod_{|Z_0|_{Y_0}} |Z''_n|_{Y_0} \rightarrow |Z_0|_{Y_0}$  is a homeomorphism. Noting that  $|Z_n| = |(Z_0)_{Y_0}|$  (and similarly for all of the other objects appearing), we conclude that  $c_{X,|V|}(Z''')$  is a universal formal Zariski colimit and it remains to show that  $c_{X,|V|}(Z''')$  has the correct functions.

Let  $m : Z \rightarrow Z'''$ ,  $m' : Z' \rightarrow Z'''$ , and  $m'' : Z'' \rightarrow Z'''$  denote the canonical morphisms, then by Theorem 2.10 we have an isomorphism of sheaves of rings  $\phi : \mathcal{O}_{Z'''} \rightarrow m'_* \mathcal{O}_{Z'} \times_{m_* \mathcal{O}_Z} m''_* \mathcal{O}_{Z''}$ . Hence, since  $m$ ,  $m'$  and  $m''$  are finite, and completion is exact on coherent modules, by [EGA, I, 10.8.8(i)] we have an isomorphism of coherent  $\mathcal{O}_{c_{X,|V|}(Z''')}$ -algebras:

$$\mathcal{O}_{c_{X,|V|}(Z''')} = (m'_* \mathcal{O}_{Z'} \times_{m_* \mathcal{O}_Z} m''_* \mathcal{O}_{Z''})^\wedge \cong (m'_* \mathcal{O}_{Z'})^\wedge \times_{(m_* \mathcal{O}_Z)^\wedge} (m''_* \mathcal{O}_{Z''})^\wedge.$$

Noting that  $(m'_* \mathcal{O}_{Z'})^\wedge \cong \widehat{m'}_* \mathcal{O}_{c_{X,|V|}(Z')}$  (and similarly for the others), we deduce that there is an isomorphism of coherent  $\mathcal{O}_{c_{X,|V|}(Z''')}$ -algebras:

$$\mathcal{O}_{c_{X,|V|}(Z''')} \cong \widehat{m'}_* \mathcal{O}_{c_{X,|V|}(Z')} \times_{\widehat{m_* \mathcal{O}_Z}} \widehat{m''_* \mathcal{O}_{c_{X,|V|}(Z'')}}.$$

It remains to show that this isomorphism is topological (where we topologize the right side with the limit topology). A general fact here is that the topology on the right is the subspace topology of the product topology on  $\widehat{m'}_* \mathcal{O}_{c_{X,|V|}(Z')} \times \widehat{m''_* \mathcal{O}_{c_{X,|V|}(Z'')}}$ . Since  $m'$ , and  $m''$  are all finite, considering the Artin-Rees Lemma [AM69, Thm. 10.11], one concludes that this is an equivalent topology to the respective adic topologies, thus we have the topological isomorphism as required.  $\square$

### 3. ADJUNCTIONS FOR QUASI-FINITE SPACES

If  $\pi : X' \rightarrow X$  is a morphism of schemes, there is a pullback functor  $\pi^* : \mathbf{QF}(X) \rightarrow \mathbf{QF}(X')$  given by  $(Z \rightarrow X) \mapsto (Z \times_X X' \rightarrow X')$ . Similarly, if  $\omega : \mathfrak{X}' \rightarrow \mathfrak{X}$  is a morphism of formal schemes, there is a pullback  $\omega^* : \mathbf{QF}(\mathfrak{X}) \rightarrow \mathbf{QF}(\mathfrak{X}')$ . In this section, we will be concerned with the construction of *left* adjoints to these functors.

**3.1. The adjoint pair  $(\pi_!, \pi^*)$ .** To motivate the construction of  $\pi_!$ , we emphasize that what we're constructing is an entirely natural thing: for a morphism of schemes  $\pi : X \rightarrow Y$ , we want to be able to take a quasi-finite and separated map  $Z \rightarrow X$  to a quasi-finite and separated map  $\pi_! Z \rightarrow Y$  in a functorial way. Two situations present themselves immediately that are worth paying closer attention to:

- (1) if  $\pi$  is quasi-finite and separated, then define  $\pi_! : \mathbf{QF}(X) \rightarrow \mathbf{QF}(Y)$  as  $\pi_!(Z \rightarrow X) = (Z \rightarrow X \xrightarrow{\pi} Y)$ . It is immediate that  $(\pi_!, \pi^*)$  is an adjoint pair.

- (2) If  $\pi$  is proper and  $s : Z \rightarrow X$  is finite, then define  $\pi_!(Z \xrightarrow{s} X)$  as  $(\text{Spec}_Y(\pi_* s_* \mathcal{O}_Z) \rightarrow Y)$ . It is clear that  $(\pi_!, \pi^*)$  forms an adjoint pair, when we restrict  $\pi^*$  to the full subcategory of  $\mathbf{QF}(Y)$  consisting of  $(Z' \xrightarrow{s'} Y)$ , where  $s'$  is finite.

For a general, quasi-finite and separated  $Z \rightarrow X$ , one can use Zariski's Main Theorem to compactify it to  $Z \hookrightarrow W \rightarrow X$ , where  $W \rightarrow X$  is finite. One then pushes  $W$  forward to get a finite  $Y$ -scheme  $W'$  and hopes that the image of  $Z$  in  $W'$  is reasonable enough to endow it with an algebraic structure, taking  $\pi_! Z$  to be this image. To get the image of  $Z$  in  $W'$  to be sufficiently well-behaved, we need to make the following definition.

**Definition 3.1.** For a morphism of schemes  $\pi : X \rightarrow Y$ , let  $\mathbf{QF}_{p.f.}(X/Y)$  denote the full subcategory of  $\mathbf{QF}(X)$  consisting of the objects  $(Z \xrightarrow{s} X)$  such that the map  $\pi \circ s : Z \rightarrow Y$  has proper fibers.

The next result gives a method of producing lots of examples of objects of  $\mathbf{QF}_{p.f.}(X/Y)$ .

**Lemma 3.2.** For a proper morphism of schemes  $\pi : X \rightarrow Y$ , and  $(Z \xrightarrow{s} Y) \in \mathbf{QF}(Y)$ , then  $(\pi^* Z \rightarrow X) \in \mathbf{QF}_{p.f.}(X/Y)$ .

*Proof.* Let  $\text{Spec } \kappa(y) \rightarrow Y$  be a point, then we form the following cartesian diagram:

$$\begin{array}{ccccc} (\pi^* Z)_y & \longrightarrow & \pi^* Z & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & X_y & & X & \\ \downarrow & & \downarrow & & \downarrow \\ Z_y & \longrightarrow & Z & & \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec } \kappa(y) & \longrightarrow & Y & & \end{array}$$

Since  $Z \rightarrow Y$  is quasi-finite, then  $Z_y \rightarrow \text{Spec } \kappa(y)$  is finite, thus  $(\pi^* Z)_y \rightarrow X_y$  is finite and since  $X_y \rightarrow \text{Spec } \kappa(y)$  is proper, we conclude that the composite  $\pi^* Z \rightarrow X \rightarrow Y$  has proper fibers.  $\square$

Recall, that a morphism of schemes  $f : X \rightarrow Y$  is **Stein** if the induced map  $f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$  is an isomorphism. We now state the main result of this section.

**Theorem 3.3.** Let  $\pi : X \rightarrow Y$  be a proper morphism of locally noetherian schemes, then there is a functor  $\pi_! : \mathbf{QF}_{p.f.}(X/Y) \rightarrow \mathbf{QF}(Y)$ , which is left adjoint to  $\pi^*$ . Moreover,

- (1) if  $(Z \xrightarrow{s} X) \in \mathbf{QF}_{p.f.}(X/Y)$ , then:
  - (a) the unit of the adjunction,  $\eta_Z : Z \rightarrow \pi^* \pi_! Z$ , is finite;
  - (b) the canonical map of schemes  $\pi^Z : Z \xrightarrow{\eta_Z} \pi^* \pi_! Z \rightarrow \pi_! Z$  is proper and Stein;
  - (c) if, in addition,  $s$  is finite, then  $\pi_!(Z \xrightarrow{s} X) = (\text{Spec}_Y(\pi_* s_* \mathcal{O}_Z) \rightarrow Y)$ ;
  - (d) if  $\pi$  is finite, then  $\pi_!(Z \xrightarrow{s} X) = (Z \xrightarrow{\pi \circ s} Y)$ .
- (2) if  $(Z' \rightarrow Y) \in \mathbf{QF}(Y)$ , then the counit of the adjunction,  $\epsilon_{Z'} : \pi_! \pi^* Z' \rightarrow Z'$ , is finite.

We defer the proof of Theorem 3.3 until the end of this section, and will presently concern ourselves with its corollaries.

The statements in Theorem 3.3 make it already consistent with our motivation, but it is possible strengthen this.

**Corollary 3.4.** *Consider a commutative diagram of locally noetherian schemes:*

$$\begin{array}{ccc} U & \xrightarrow{g} & X \\ \rho \downarrow & & \downarrow \pi \\ V & \xrightarrow{f} & Y \end{array}$$

Suppose  $\pi, \rho$  are proper, then

- (1) if  $f, g$  are proper, there are natural isomorphisms of functors from  $\mathbf{QF}_{p.f.}(U/Y)$  to  $\mathbf{QF}(Y)$ :

$$f_! \rho_! \implies (f \circ \rho)_! \iff (\pi \circ g)_! \longleftarrow \pi_! g_!;$$

- (2) if  $f, g$  are instead quasi-finite and separated, there is a natural isomorphism of functors from  $\mathbf{QF}_{p.f.}(U/Y)$  to  $\mathbf{QF}(Y)$ :

$$\pi_! g_! \implies f_! \rho_!.$$

Given a quasi-compact and quasi-separated morphism of schemes  $g : U \rightarrow V$ , then it has a **Stein factorization**:  $U \xrightarrow{g'} \mathrm{Spec}_V g_* \mathcal{O}_U \rightarrow V$ , where  $g'$  is a Stein morphism. One obtains immediately from Corollary 3.4(2):

**Corollary 3.5.** *If  $\pi : X \rightarrow Y$  is a proper morphism of locally noetherian schemes,  $(Z' \rightarrow Y) \in \mathbf{QF}(Y)$ , set  $Z := \pi^* Z'$  and let  $\pi_{Z'} : Z \rightarrow Z'$  denote the induced map. There is a natural  $Z'$ -isomorphism  $(\pi_{Z'})_! Z \rightarrow \pi_! Z$ . Thus,  $Z \xrightarrow{\pi^Z} \pi_! \pi^* Z' \xrightarrow{\epsilon_{Z'}} Z'$  is canonically isomorphic to the Stein factorization of the morphism  $\pi_{Z'} : Z \rightarrow Z'$ .*

The next result is important for technical reasons, as it allows one to work flat locally.

**Corollary 3.6.** *Consider a commutative diagram of locally noetherian schemes,*

$$\begin{array}{ccc} X' & \xrightarrow{p'} & X \\ \pi' \downarrow & & \downarrow \pi \\ Y' & \xrightarrow{p} & Y \end{array}$$

then if  $\pi$  and  $\pi'$  are proper, or quasi-finite and separated there is a natural transformation:

$$\Delta : \pi'_! p'^* \implies p^* \pi_!.$$

If the diagram is cartesian, and

- (1) if  $\pi$  is proper, then  $\Delta$  induces a finite, universal homeomorphism; or
- (2) if  $p$  is flat, then  $\Delta$  is an isomorphism of functors; or
- (3) if  $\pi$  is quasi-finite and separated, then  $\Delta$  is an isomorphism of functors.

Combined with the étale-local structure theorem for quasi-finite morphisms [EGA, IV, 18.12.1] and Corollary 3.6, the following result allows one to compute the adjunction map explicitly.

**Corollary 3.7.** *If  $\pi : X \rightarrow Y$  is a proper map of locally noetherian schemes,  $(Z \xrightarrow{s} Y) \in \mathbf{QF}(Y)$ , and  $U \rightarrow Z$  is a quasi-finite, separated, and flat morphism (e.g. an open immersion), then there is*



a cartesian diagram:

$$\begin{array}{ccc} \pi_! \pi^* \mathcal{U} & \xrightarrow{\eta_{\mathcal{U}}} & \mathcal{U} \\ \downarrow & & \downarrow \\ \pi_! \pi^* \mathcal{Z} & \xrightarrow{\eta} & \mathcal{Z} \end{array}$$

In the case that the induced  $s_{\mathcal{U}} : \mathcal{U} \rightarrow Y$  is finite, we obtain a canonical isomorphism

$$\pi_! \pi^* \mathcal{U} \cong \mathrm{Spec}_Y(\pi_* \pi^*(s_{\mathcal{U}})_* \mathcal{O}_{\mathcal{U}}).$$

In this case, the map  $\eta_{\mathcal{U}} : \pi_! \pi^* \mathcal{U} \rightarrow \mathcal{U}$  is given by the adjunction map on coherent sheaves of  $\mathcal{O}_Y$ -algebras:  $(s_{\mathcal{U}})_* \mathcal{O}_{\mathcal{U}} \rightarrow \pi_* \pi^*(s_{\mathcal{U}})_* \mathcal{O}_{\mathcal{U}}$ .

Before we prove Theorem 3.3, we include here for future reference a technical lemma that will be used frequently. First, we observe that if a quasi-compact and quasi-separated morphism of schemes  $f : X \rightarrow Y$  is Stein, then it remains so after flat base change on  $Y$ . If the morphism  $f$  is a proper and Stein morphism of locally noetherian schemes, then it is surjective and has geometrically connected fibers. Note that even though the property of being Stein is not preserved under arbitrary change of base, the properties of surjectivity and geometric connectivity of the fibers are.

**Lemma 3.8.** *Consider a commutative diagram of schemes*

$$\begin{array}{ccc} & X & \\ f \swarrow & & \searrow g \\ Y & \xrightarrow{h} & Z \end{array}$$

where  $f, g$  are quasi-compact, quasi-separated, surjective and have geometrically connected fibers. If

- (1)  $h$  is integral; or
- (2)  $f, g$  are universally closed, and  $h$  has discrete fibers,

then  $h$  is an integral, universal homeomorphism. If, in addition,  $f, g$  are Stein, then  $h$  is an isomorphism.

*Proof.* Under both assumptions, we obtain that  $h$  is surjective, and has geometrically connected fibers. Since in either case, the fibers of  $h$  are discrete,  $h$  is radiciel. Furthermore,  $h$  is universally closed. By [EGA, IV, 18.12.10–11],  $h$  is an integral universal homeomorphism, which proves the first claim. To prove the latter claim, it suffices to check that the map  $h^\# : \mathcal{O}_Z \rightarrow h_* \mathcal{O}_Y$  is an isomorphism of sheaves on  $|Z|$ . We may conclude the proof by observing that since the maps  $f, g$  are Stein, the map  $h^\#$  factors as the following composition of isomorphisms:

$$\mathcal{O}_Z \cong g_* \mathcal{O}_X \cong (h \circ f)_* \mathcal{O}_X \cong h_* f_* \mathcal{O}_X \cong h_* \mathcal{O}_Y. \quad \square$$

With this result at our disposal, we return to the task at hand.

*Proof of Theorem 3.3.* Let  $s : Z \rightarrow X$  belong to  $\mathbf{QF}_{p.f.}(X/Y)$  and let  $\mathcal{B}_Z$  be the integral closure of  $\mathcal{O}_X$  in  $s_* \mathcal{O}_Z$ . We set  $\bar{Z} = \mathrm{Spec}_X \mathcal{B}_Z$ , and call this the **universal  $X$ -compactification** of  $Z$ , for reasons that will we make clear now. Zariski's Main Theorem [EGA, IV, 18.12.13] shows that  $j : Z \rightarrow \bar{Z}$  is an open immersion and moreover, that there is a finite  $X$ -scheme  $t : W \rightarrow X$  with  $\iota : \bar{Z} \rightarrow W$  such that  $\iota \circ j : Z \rightarrow W$  is an open immersion. We may further assume that  $t_* \mathcal{O}_W \subset \mathcal{B}_Z$ .

Let  $\bar{Z}' = \operatorname{Spec}_Y \pi_* \mathcal{B}_Z$ ,  $W' = \operatorname{Spec}_Y \pi_* t_* \mathcal{O}_W$ . Note that there is a canonical map  $\iota' : \bar{Z}' \rightarrow W'$  given by the inclusion of sheaves  $\pi_* t_* \mathcal{O}_W \subset \pi_* \mathcal{B}_Z$  and we have the following commutative diagram:

$$\begin{array}{ccccc} \bar{Z} & \xrightarrow{\iota} & W & \xrightarrow{t} & X \\ p \downarrow & & q \downarrow & & \downarrow \pi \\ \bar{Z}' & \xrightarrow{\iota'} & W' & \xrightarrow{t'} & Y \end{array}.$$

Note that by construction,  $p$  and  $q$  are Stein morphisms. Since  $\pi$  is proper, by [EGA, III, 3.2.1],  $t'$  is finite, and  $q$  is proper with geometrically connected fibers. One also concludes that  $p$  is quasi-compact, separated, universally closed, and surjective. In particular, both  $p$  and  $q$  are universally submersive. Furthermore, it follows that  $\iota'$  is universally closed and affine, thus by [EGA, IV, 18.12.8], it is integral.

Let  $|Z'| \subset |W'|$  denote the set of points of  $W'$  which are the set-theoretic image of  $|Z|$  under  $q$ . We claim that  $|Z'|$  is open. Since  $q$  is universally submersive, it suffices to show that  $q^{-1}|Z'|$  is open in  $|W|$ . Let  $w' \in |W'|$ , then  $Z_{w'} \rightarrow W_{w'}$  is an open immersion by construction (where  $W_{w'}$  is the fiber of  $W \rightarrow W'$ ). Also, since  $Z_{w'}$  is proper over  $\{w'\}$ , then  $Z_{w'} \rightarrow W_{w'}$  is closed ( $W_{w'}$  is separated). Hence,  $Z_{w'} \rightarrow W_{w'}$  is open and closed, but  $W_{w'}$  is connected and hence we conclude that either  $Z_{w'} = \emptyset$  or  $Z_{w'} = W_{w'}$ . That is,  $q^{-1}|Z'| = |Z|$  which shows that  $|Z'|$  is an open subset of  $|W'|$  and we define  $Z'$  to be the open subscheme of  $W'$  corresponding to the open set  $|Z'|$ . In particular, we observe that  $Z' \times_{W'} W = q^{-1}(Z') = Z \subset W$  and so  $q|_Z : Z \rightarrow Z'$  is proper and Stein. Moreover, since  $j$  is a dense open immersion and  $\iota$  is separated, then,  $|Z| = \iota^{-1}\iota(|Z|)$ . Thus, as  $p, q, \iota, \iota'$  are surjective, we have equalities of subsets of  $|\bar{Z}|$ :

$$|Z| = \iota^{-1}\iota(|Z|) = \iota^{-1}q^{-1}qt(|Z|) = p^{-1}\iota'^{-1}\iota'p(|Z|).$$

Hence,

$$p(|Z|) = \iota'^{-1}\iota'p(|Z|) \implies p^{-1}p(|Z|) = |Z|.$$

Thus, since  $p$  is universally submersive, we conclude that  $p(|Z|)$  is an open subscheme of  $|\bar{Z}'|$ . We define  $\pi_! Z$  to be the open subscheme of  $\bar{Z}'$  corresponding to the open subset  $p(|Z|)$ . Note, that there is a natural map  $\pi^Z : Z \rightarrow \pi_! Z$  which is universally closed, separated, surjective, and Stein. Moreover, since the map  $\pi_! Z \rightarrow Z'$  is integral, we may apply Lemma 3.8 to conclude that the map  $\pi_! Z \rightarrow Z'$  is an isomorphism of  $Y$ -schemes. Thus,  $(\pi_! Z \rightarrow Y) \in \mathbf{QF}(Y)$  and  $\pi^Z : Z \rightarrow \pi_! Z$  is proper and Stein.

To define a functor  $\pi_!$ , we must next prove that for any map  $f : Z \rightarrow Z^0$  in  $\mathbf{QF}_{p.f.}(X/Y)$ , there is a map  $\pi_! f : \pi_! Z \rightarrow \pi_! Z^0$  such that if  $g : Z^0 \rightarrow Z^1$  is any other map in  $\mathbf{QF}_{p.f.}(X/Y)$ , then  $\pi_!(g \circ f) = (\pi_! g) \circ (\pi_! f)$  as maps in  $\mathbf{QF}(Y)$ .

So, let  $s^0 : Z^0 \rightarrow X$  belong to  $\mathbf{QF}_{p.f.}(X/Y)$  and suppose that we have a morphism  $f : Z \rightarrow Z^0$  in  $\mathbf{QF}_{p.f.}(X/Y)$ . Retaining similar notation used at the beginning of the proof, we have an open immersion  $Z^0 \hookrightarrow \bar{Z}^0$ , where  $\bar{Z}^0 = \operatorname{Spec}_X \mathcal{B}_{Z^0}$  and  $\mathcal{B}_{Z^0}$  is the integral closure of  $\mathcal{O}_X$  in  $(s^0)_* \mathcal{O}_{Z^0}$ . First of all, we observe that the morphism  $f : Z \rightarrow Z^0$  provides a map of sheaves  $\tilde{f} : (s^0)_* \mathcal{O}_{Z^0} \rightarrow s_* \mathcal{O}_Z$ . In particular, since  $\mathcal{B}_{Z^0}$  is the integral closure of  $\mathcal{O}_X$  in  $(s^0)_* \mathcal{O}_{Z^0}$ , then the image of any section of  $\mathcal{B}_{Z^0}$  in  $s_* \mathcal{O}_Z$  under  $\tilde{f}$  is integral over  $\mathcal{O}_X$  and *a fortiori* lies in  $\mathcal{B}_Z$ . Consequently, the map  $\tilde{f}|_{\mathcal{B}_{Z^0}} : \mathcal{B}_{Z^0} \rightarrow s_* \mathcal{O}_Z$  factors through  $\mathcal{B}_Z$ . Hence, associated to any  $f : Z \rightarrow Z^0$  in  $\mathbf{QF}_{p.f.}(X/Y)$ , there is a natural map  $\bar{f} : \bar{Z} \rightarrow \bar{Z}^0$ , which is integral.

Pushing all of this forward to  $Y$  by the morphism  $\pi$ , we obtain a natural map  $\bar{f}' : \bar{Z}' \rightarrow \bar{Z}^0$  and the claim is that this restricts to a map  $\pi_! f : \pi_! Z \rightarrow \pi_! Z^0$ . But considering that  $\pi_! Z$  (resp.  $\pi_! Z^0$ ) is the set-theoretic image of  $Z$  in  $\bar{Z}'$  (resp.  $\bar{Z}^0$ ), then this is certainly true on the level of sets, and since  $\pi_! Z$  (resp.  $\pi_! Z^0$ ) are open subschemes of  $\bar{Z}'$  (resp.  $\bar{Z}^0$ ), then this is clear. In particular, we can see by construction that if  $g : Z^0 \rightarrow Z^1$  is another morphism in  $\mathbf{QF}_{p.f.}(X/Y)$ , then  $\pi_!(g \circ f) = (\pi_! g) \circ (\pi_! f)$ . Hence, we have defined a functor  $\pi_! : \mathbf{QF}_{p.f.}(X/Y) \rightarrow \mathbf{QF}(Y)$ .

We now proceed to prove that  $(\pi_!, \pi^*)$  is an adjoint pair. For this, we construct a pair of natural transformations  $\eta : \mathrm{id}_{\mathbf{QF}_{p.f.}(X/Y)} \Rightarrow \pi^* \pi_!$ ,  $\epsilon : \pi_! \pi^* \Rightarrow \mathrm{id}_{\mathbf{QF}(Y)}$  such that  $\epsilon(\pi_!) \circ \pi_!(\eta) = \mathrm{id}_{\pi_!}$  and  $(\pi^* \epsilon) \circ \eta(\pi^*) = \mathrm{id}_{\pi^*}$ .

Let  $Z \rightarrow X$  belong to  $\mathbf{QF}_{p.f.}(X/Y)$ , then by construction there is a map  $\pi^Z : Z \rightarrow \pi_! Z$ , and thus a map  $\eta_Z : Z \rightarrow \pi^* \pi_! Z$ . In particular, this construction is natural in  $Z$  and so we have defined  $\eta : \mathrm{id}_{\mathbf{QF}_{p.f.}(X/Y)} \Rightarrow \pi^* \pi_!$ . Note that since  $Z \rightarrow \pi_! Z$  and  $\pi^* \pi_! Z \rightarrow \pi_! Z$  are proper, and  $\eta_Z : Z \rightarrow \pi^* \pi_! Z$  is quasi-finite and separated, then by [EGA, IV, 18.12.4], it is finite.

Now suppose that  $Z' \rightarrow Y$  belongs to  $\mathbf{QF}(Y)$ , and let  $\bar{Z}' \rightarrow Y$  be the universal  $Y$ -compactification of  $Z'$ , then if  $(\pi^* \bar{Z}')$  is the universal  $X$ -compactification of  $\pi^* Z'$ , there is a unique map  $(\pi^* \bar{Z}') \rightarrow \bar{Z}' \times_Y X$ . Pushing this all back down to  $Y$  and using adjunction for sheaves again, we obtain a natural map  $\pi_! \pi^* Z' \rightarrow \bar{Z}'$ , but this map factors through  $Z' \hookrightarrow \bar{Z}'$  and so we have constructed a natural transformation  $\epsilon : \pi_! \pi^* \Rightarrow \mathrm{id}_{\mathbf{QF}(Y)}$ . Similarly, we observe that the map  $\epsilon_{Z'} : \pi_! \pi^* Z' \rightarrow Z'$  is quasi-finite and separated, and the maps  $\pi^* Z' \rightarrow \pi_! \pi^* Z'$  and  $\pi^* Z' \rightarrow Z'$  are proper. Again, by [EGA, IV, 18.12.4], we conclude that the map  $\epsilon_{Z'} : \pi_! \pi^* Z' \rightarrow Z'$  is finite.

It remains to check the unit/counit equations. Note that the first equation amounts to checking that

$$\pi_! Z \xrightarrow{\pi_!(\eta_Z)} \pi_!(\pi^* \pi_! Z) \xrightarrow{\epsilon_{\pi_! Z}} \pi_! Z$$

is the identity map. Since  $\epsilon$  and  $\eta$  are constructed via adjunction of sheaves and then restricted to open subschemes, then it is clear that this equation holds. One may argue similarly for the other equation.  $\square$

*Proof of Corollary 3.4.* The proof of (1) is immediate from the functoriality of pullbacks and the universal properties defining adjoints. For (2), we fix  $(Z \xrightarrow{s} U) \in \mathbf{QF}_{p.f.}(U/V) = \mathbf{QF}_{p.f.}(U/Y)$ . By Theorem 3.3(1b), there is a natural map  $\rho^Z : Z \rightarrow \rho_! Z$ . As  $f$  is quasi-finite and separated, we obtain a natural map  $\phi : Z \rightarrow f_! \rho_! Z$ , which is proper and Stein. Also, since the composition of  $\phi$  with the structure map of  $\rho_! Z$  over  $Y$  agrees with the composition  $Z \xrightarrow{g} X \xrightarrow{\pi} Y$ , there is a naturally induced morphism  $Z \rightarrow g^* \pi^* f_! \rho_! Z$ . By adjunction, there is thus a natural map  $\psi : \pi_! g_! Z \rightarrow f_! \rho_! Z$ , which is quasi-finite, and separated. In particular, we have the commutative triangle:

$$\begin{array}{ccc} & Z & \\ \pi^{g_! Z} \swarrow & & \searrow \phi \\ \pi_! g_! Z & \xrightarrow{\psi} & f_! \rho_! Z \end{array} .$$

By Theorem 3.3(1b), the map  $\pi^{g_! Z}$  is proper and Stein, thus by Lemma 3.8,  $\psi$  is an isomorphism of schemes.  $\square$

*Proof of Corollary 3.6.* The existence of the natural transformation is a standard property of adjoints, so we omit the proof. Since (3) is obvious, we concentrate on (1) and (2). So, let  $(Z \xrightarrow{s} X)$ , then we obtain from this data a commutative diagram:

$$\begin{array}{ccccc}
 p'^*Z & \xrightarrow{\quad} & Z & & \\
 \swarrow & & \searrow & & \searrow \\
 & & X' & \xrightarrow{p'} & X \\
 \swarrow & & \downarrow & & \downarrow \\
 \pi'_! p'^*Z & \xrightarrow{\quad} & p^* \pi_! Z & \xrightarrow{\quad} & \pi_! Z \\
 \swarrow & & \downarrow & & \downarrow \\
 & & Y' & \xrightarrow{p} & Y
 \end{array}$$

where the bottom, top, front and back squares are cartesian. The map  $\pi'_! p'^*Z \rightarrow p^* \pi_! Z$  is quasi-finite (thus has discrete fibers), and by Theorem 3.3(1b), the map  $p'^*Z \rightarrow \pi'_! p'^*Z$  is proper and Stein. Also, by Theorem 3.3(1b),  $p'^*Z \rightarrow p^* \pi_! Z$  is proper and has geometrically connected fibers (if  $p$  is flat, this map is in addition Stein), as it is the base change (resp. flat base change) of the proper and Stein map  $Z \rightarrow \pi_! Z$ . By Lemma 3.8, the map  $\pi'_! p'^*Z \rightarrow p^* \pi_! Z$  is a finite, universal homeomorphism (resp. isomorphism).  $\square$

*Remark 3.9.* Theorem 3.3 and its corollaries readily extend to representable morphisms of algebraic stacks. Using the results of Faltings [Fal03], Olsson [Ols05], and Conrad [Con] it is also possible to extend these results to non-representable morphisms of algebraic stacks.

**3.2. The adjoint pair  $(\omega_!, \omega^*)$ .** As in §3.1, for a quasi-finite, separated, and adic morphism  $\omega : \mathfrak{X} \rightarrow \mathfrak{Y}$  of locally noetherian formal schemes, the functor  $\omega_! : \mathbf{QF}(\mathfrak{X}) \rightarrow \mathbf{QF}(\mathfrak{Y})$  which sends  $(\mathfrak{Z} \xrightarrow{\sigma} \mathfrak{X})$  to  $(\mathfrak{Z} \xrightarrow{\omega \circ \sigma} \mathfrak{Y}) \in \mathbf{QF}(\mathfrak{Y})$  is left adjoint to  $\omega^*$ . The purpose of this section is to construct, for a proper and adic morphism of locally noetherian formal schemes  $\omega : \mathfrak{X} \rightarrow \mathfrak{Y}$ , a left adjoint to  $\omega^*$ , and extend all of the results of §3.1 to locally noetherian formal schemes. In particular, we start with the following

**Definition 3.10.** Let  $\mathfrak{X}$  be a locally noetherian formal scheme. For an adic morphism of locally noetherian formal schemes  $\omega : \mathfrak{X} \rightarrow \mathfrak{Y}$ , let  $\mathbf{QF}_{p.f.}(\mathfrak{X}/\mathfrak{Y})$  denote the full subcategory of  $\mathbf{QF}(\mathfrak{X})$  consisting of those objects  $(\mathfrak{Z} \xrightarrow{\sigma} \mathfrak{X})$  with the composition  $\mathfrak{Z} \xrightarrow{\sigma} \mathfrak{X} \xrightarrow{\omega} \mathfrak{Y}$  having proper fibers.

There is a formal analog of Lemma 3.2, which says that if  $\omega : \mathfrak{X} \rightarrow \mathfrak{Y}$  is adic and proper, and  $(\mathfrak{Z}' \rightarrow \mathfrak{Y}) \in \mathbf{QF}(\mathfrak{Y})$ , then  $\omega^*(\mathfrak{Z}' \rightarrow \mathfrak{Y}) \in \mathbf{QF}_{p.f.}(\mathfrak{X}/\mathfrak{Y})$ .

We say that a map of locally noetherian formal schemes  $f : \mathfrak{X} \rightarrow \mathfrak{Y}$  is **Stein** if the induced map  $\mathcal{O}_{\mathfrak{Y}} \rightarrow f_* \mathcal{O}_{\mathfrak{X}}$  is a topological isomorphism.

**Example 3.11.** If  $\mathfrak{Y}$  is a noetherian formal scheme, let  $\mathcal{I}$  be an ideal of definition. Consider an adic and proper morphism  $\omega : \mathfrak{X} \rightarrow \mathfrak{Y}$  and let  $\omega_n : X_n \rightarrow Y_n$  denote the proper morphism of schemes induced by pullback of  $f$  by the morphism  $Y_n := V(\mathcal{I}^n) \rightarrow \mathfrak{Y}$ . If  $\omega_n$  is Stein for all  $n$ , then  $\omega$  is Stein. Indeed, by [EGA, III, 3.4.4] there is a topological isomorphism

$$\omega_* \mathcal{O}_{\mathfrak{X}} \cong \varprojlim_n (\omega_n)_* \mathcal{O}_{X_n} \cong \varprojlim_n \mathcal{O}_{Y_n} \cong \mathcal{O}_{\mathfrak{Y}}.$$

Note that it does not follow that if  $\omega : \mathfrak{X} \rightarrow \mathfrak{Y}$  is an adic, proper, and Stein morphism, then  $\omega_n : X_n \rightarrow Y_n$  is Stein. We now have the formal analog of Theorem 3.3.

**Theorem 3.12.** *Let  $\omega : \mathfrak{X} \rightarrow \mathfrak{Y}$  be a proper, adic, morphism of locally noetherian formal schemes. There is a functor  $\omega_! : \mathbf{QF}_{p.f.}(\mathfrak{X}/\mathfrak{Y}) \rightarrow \mathbf{QF}(\mathfrak{Y})$ , which is left adjoint to  $\omega^*$ . Moreover,*

- (1) *if  $(\mathfrak{Z} \xrightarrow{\sigma} \mathfrak{X}) \in \mathbf{QF}_{p.f.}(\mathfrak{X}/\mathfrak{Y})$ , then:*
  - (a) *the unit of the adjunction,  $\eta_{\mathfrak{Z}} : \mathfrak{Z} \rightarrow \omega^* \omega_! \mathfrak{Z}$ , is finite;*
  - (b) *the canonical map of locally noetherian formal schemes  $\omega^{\mathfrak{Z}} : \mathfrak{Z} \xrightarrow{\eta_{\mathfrak{Z}}} \omega^* \omega_! \mathfrak{Z} \rightarrow \omega_! \mathfrak{Z}$  is proper, adic, and Stein;*
  - (c) *if, in addition,  $\sigma$  is finite, then  $\omega_!(\mathfrak{Z} \xrightarrow{\sigma} \mathfrak{X}) = (\mathrm{Spf}_{\mathfrak{Y}}(\omega_* \sigma_* \mathcal{O}_{\mathfrak{Z}}) \rightarrow \mathfrak{Y})$ ;*
  - (d) *if  $\omega$  is finite, then  $\omega_!(\mathfrak{Z} \xrightarrow{\sigma} \mathfrak{X}) = (\mathfrak{Z} \xrightarrow{\omega \circ \sigma} \mathfrak{Y})$ .*
- (2) *if  $(\mathfrak{Z}' \rightarrow \mathfrak{Y}) \in \mathbf{QF}(\mathfrak{Y})$ , then the counit of the adjunction,  $\epsilon_{\mathfrak{Z}'} : \omega_! \omega^* \mathfrak{Z}' \rightarrow \mathfrak{Z}'$ , is finite.*

As in §3.1, we will defer the proof of Theorem 3.12 until the end of this section, and focus on the implications.

**Corollary 3.13.** *Consider a commutative diagram of locally noetherian formal schemes:*

$$\begin{array}{ccc} \mathfrak{U} & \xrightarrow{g} & \mathfrak{X} \\ \rho \downarrow & & \downarrow \omega \\ \mathfrak{Y} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

*Suppose  $\omega, \rho$  are adic and proper, then*

- (1) *if  $f, g$  are adic and proper, there are natural isomorphisms of functors from  $\mathbf{QF}_{p.f.}(\mathfrak{U}/\mathfrak{Y})$  to  $\mathbf{QF}(\mathfrak{Y})$ :*

$$f_! \rho_! \implies (f \circ \rho)_! \iff (\omega \circ g)_! \iff \omega_! g_!;$$

- (2) *if  $f, g$  are instead adic, quasi-finite, and separated, there is a natural isomorphism of functors from  $\mathbf{QF}_{p.f.}(\mathfrak{U}/\mathfrak{Y})$  to  $\mathbf{QF}(\mathfrak{Y})$ :*

$$\omega_! g_! \implies f_! \rho_!.$$

**Corollary 3.14.** *If  $\omega : \mathfrak{X} \rightarrow \mathfrak{Y}$  is a proper morphism of locally noetherian formal schemes,  $(\mathfrak{Z}' \rightarrow \mathfrak{Y}) \in \mathbf{QF}(\mathfrak{Y})$ , set  $\mathfrak{Z} := \omega^* \mathfrak{Z}'$  and let  $\omega_{\mathfrak{Z}'} : \mathfrak{Z} \rightarrow \mathfrak{Z}'$  denote the induced map. There is a natural  $\mathfrak{Z}'$ -isomorphism  $(\omega_{\mathfrak{Z}'})_! \mathfrak{Z} \rightarrow \omega_! \mathfrak{Z}$ . Thus,*

$$\mathfrak{Z} \xrightarrow{\omega^{\mathfrak{Z}}} \omega_! \omega^* \mathfrak{Z}' \xrightarrow{\epsilon_{\mathfrak{Z}'}} \mathfrak{Z}'$$

*is canonically isomorphic to the Stein factorization of the morphism  $\omega_{\mathfrak{Z}'} : \mathfrak{Z} \rightarrow \mathfrak{Z}'$ .*

**Corollary 3.15.** *Consider a commutative diagram of locally noetherian formal schemes:*

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{p'} & \mathfrak{X} \\ \omega' \downarrow & & \downarrow \omega \\ \mathfrak{Y}' & \xrightarrow{p} & \mathfrak{Y} \end{array}$$

*then if  $\omega$  and  $\omega'$  are adic and proper, or adic, quasi-finite, and separated, there is a natural transformation:*

$$\Delta : \omega'_! p'^* \implies p^* \omega_!.$$

*If the diagram is cartesian, and*

- (1) if  $\omega$  is proper, then  $\Delta$  induces an adic, finite, universal homeomorphism; or
- (2) if  $\mathfrak{p}$  is adic and flat, then  $\Delta$  is an isomorphism of functors; or
- (3) if  $\omega$  is adic, quasi-finite, and separated, then  $\Delta$  is an isomorphism of functors.

**Corollary 3.16.** *If  $\omega : \mathfrak{X} \rightarrow \mathfrak{Y}$  is a proper and adic map of locally noetherian formal schemes,  $(\mathfrak{Z} \xrightarrow{\sigma} \mathfrak{Y}) \in \mathbf{QF}(\mathfrak{Y})$ , and  $\mathfrak{U} \rightarrow \mathfrak{Z}$  is an adic, quasi-finite flat morphism, then we have a cartesian diagram:*

$$\begin{array}{ccc} \omega_! \omega^* \mathfrak{U} & \xrightarrow{\eta_{\mathfrak{U}}} & \mathfrak{U} \\ \downarrow & & \downarrow \\ \omega_! \omega^* \mathfrak{Z} & \xrightarrow{\eta} & \mathfrak{Z} \end{array}$$

If the composition  $\sigma_{\mathfrak{U}} : \mathfrak{U} \rightarrow \mathfrak{Y}$  is finite, there is a canonical isomorphism

$$\omega_! \omega^* \mathfrak{U} \cong \mathrm{Spf}_{\mathfrak{Y}}(\omega_* \omega^*(\sigma_{\mathfrak{U}})_* \mathcal{O}_{\mathfrak{U}})$$

such that the map  $\eta_{\mathfrak{U}}$  corresponds to the adjunction map on coherent sheaves of  $\mathcal{O}_{\mathfrak{Y}}$ -algebras:  $(\sigma_{\mathfrak{U}})_* \mathcal{O}_{\mathfrak{U}} \rightarrow \omega_* \omega^*(\sigma_{\mathfrak{U}})_* \mathcal{O}_{\mathfrak{U}}$ .

We now return to the task of proving Theorem 3.12.

**Lemma 3.17.** *Let  $\mathfrak{X}$  be a locally noetherian formal scheme and let  $\mathfrak{U} \rightarrow \mathfrak{X}$  be an adic, locally quasi-finite, and separated morphism, suppose that  $[\mathfrak{R} \rightrightarrows \mathfrak{U}]$  is an adic fppf equivalence relation over  $\mathfrak{X}$  such that  $\mathfrak{R} \rightarrow \mathfrak{U} \times_{\mathfrak{X}} \mathfrak{U}$  is a closed immersion, then this equivalence relation has a uniform formal geometric quotient  $\mathfrak{Z}$  which is adic, locally quasi-finite, and separated over  $\mathfrak{X}$ .*

*Proof.* Let  $\mathcal{I}$  be an ideal of definition for  $\mathfrak{X}$  and denote  $X_{\mathcal{I}} = V(\mathcal{I})$ ,  $U_{\mathcal{I}} = \mathfrak{U} \times_{\mathfrak{X}} X_{\mathcal{I}}$ , and  $R_{\mathcal{I}} = \mathfrak{R} \times_{\mathfrak{X}} X_{\mathcal{I}}$ . Then, the hypotheses ensure that  $[R_{\mathcal{I}} \rightrightarrows U_{\mathcal{I}}]$  is an fppf equivalence relation over  $X_{\mathcal{I}}$ . We observe that the quotient of this fppf equivalence relation in the category of algebraic spaces is a *scheme*  $Z_{\mathcal{I}}$ , as it is locally quasi-finite and separated over  $X_{\mathcal{I}}$ . This quotient is also a uniform geometric quotient, and regarding the objects as locally noetherian formal schemes one sees that it is, in fact, a uniform formal geometric quotient. Note that for any other ideal of definition  $\mathcal{J} \supset \mathcal{I}$  there is a natural isomorphism  $Z_{\mathcal{I}} \times_{\mathfrak{X}} X_{\mathcal{J}} \cong Z_{\mathcal{J}}$ , thus the directed system  $\{Z_{\mathcal{I}}\}_{\mathcal{I}}$  is adic over  $\mathfrak{X}$ . Taking the colimit of this system in the category of topologically ringed spaces, produces a locally noetherian formal scheme  $\mathfrak{Z}$ , which is adic, locally quasi-finite, and separated over  $\mathfrak{X}$ , and is a uniform formal geometric quotient of the given equivalence relation.  $\square$

*Proof of Theorem 3.12.* Let  $\mathcal{I}$  be an ideal of definition for  $\mathfrak{Y}$ , let  $Y_{\mathcal{I}} = V(\mathcal{I})$ , then  $Y_{\mathcal{I}}$  is a locally noetherian scheme and we take  $X_{\mathcal{I}} = \mathfrak{X} \times_{\mathfrak{Y}} Y_{\mathcal{I}}$ , and since  $\omega$  is adic,  $\omega_{\mathcal{I}} : X_{\mathcal{I}} \rightarrow Y_{\mathcal{I}}$  is a proper morphism of schemes. If  $\mathcal{J} \subset \mathcal{I}$  is another ideal of definition we obtain a commutative square:

$$\begin{array}{ccc} X_{\mathcal{I}} & \xrightarrow{\iota'_{\mathcal{J}}} & X_{\mathcal{J}} \\ \omega_{\mathcal{I}} \downarrow & & \downarrow \omega_{\mathcal{J}} \\ Y_{\mathcal{I}} & \xrightarrow{\iota_{\mathcal{J}}} & Y_{\mathcal{J}} \end{array}$$

Let  $(\mathfrak{Z} \rightarrow \mathfrak{X})$  belong to  $\mathbf{QF}_{\mathrm{p.f.}}(\mathfrak{X})$ , then we observe that by Corollary 3.6, there is a natural, finite universal homeomorphism:

$$\mathfrak{J}_{\mathcal{J}} : (\omega_{\mathcal{I}})_! Z_{\mathcal{I}} \longrightarrow (\iota_{\mathcal{J}})^*(\omega_{\mathcal{J}})_! Z_{\mathcal{J}}.$$

Let  $\mathcal{K}$  denote the largest ideal of definition of  $\mathfrak{Y}$  and for any other ideal of definition  $\mathcal{I}$  of  $\mathfrak{Y}$ , let  $\mathcal{J} = \mathcal{I}\mathcal{K}$ , then by the above, the map  $\mathcal{J}$  is a homeomorphism. Define the topologically ringed space  $\omega_{\mathcal{I}}\mathfrak{Z}$  to be the weak formal geometric colimit of the directed system  $\{(\omega_{\mathcal{I}})_! Z_{\mathcal{I}}\}_{\mathcal{I}}$ , and note that we are yet to show that  $\omega_{\mathcal{I}}\mathfrak{Z}$  is a formal scheme. Explicitly:

$$\omega_{\mathcal{I}}\mathfrak{Z} = (|(\omega_{\mathcal{K}})_! Z_{\mathcal{K}}|, \varinjlim_{\mathcal{J}} \mathcal{O}_{(\omega_{\mathcal{I}})_! Z_{\mathcal{I}}}),$$

with the topology on the sheaf of rings given by the induced limit topology. There is also a unique morphism of topologically ringed spaces  $t : \omega_{\mathcal{I}}\mathfrak{Z} \rightarrow \mathfrak{Y}$ . We will show that  $\omega_{\mathcal{I}}\mathfrak{Z} \in \mathbf{QF}(\mathfrak{Y})$  for any  $\mathfrak{Z}$ . To prove that  $\omega_{\mathcal{I}}\mathfrak{Z}$  is a formal scheme, it does not suffice to apply [EGA, I, 10.6.3], as the directed system defining  $\omega_{\mathcal{I}}\mathfrak{Z}$  is not, in general, a sequence of nilimmersions.

Fix  $y \in |\mathfrak{Y}| = |Y_{\mathcal{K}}|$ . By [EGA, IV, 18.12.1], there is an étale neighborhood  $(\tilde{Y}_{\mathcal{K},y}, \tilde{y}_{\mathcal{K}}) \rightarrow (Y_{\mathcal{K}}, y)$  such that  $((\omega_{\mathcal{K}})_! Z_{\mathcal{K}}) \times_{Y_{\mathcal{K}}} \tilde{Y}_{\mathcal{K},y}$  has an open and closed subscheme  $U_{\mathcal{K},y}$  containing the fiber over  $\tilde{y}$  such that  $U_{\mathcal{K},y} \rightarrow \tilde{Y}_{\mathcal{K},y}$  is finite. Using [EGA, IV, 18.1.2], we see that for any ideal of definition  $\mathcal{I}$  of  $\mathfrak{Y}$ , one produces a unique étale morphism  $(\tilde{Y}_{\mathcal{I},y}, \tilde{y}_{\mathcal{I}}) \rightarrow (Y_{\mathcal{I}}, y)$  lifting the morphism  $(\tilde{Y}_{\mathcal{K},y}, \tilde{y}_{\mathcal{K}}) \rightarrow (Y_{\mathcal{K}}, y)$ . In particular, one obtains a unique, open and closed subscheme  $U_{\mathcal{I},y} \hookrightarrow ((\omega_{\mathcal{I}})_! Z_{\mathcal{I}}) \times_{Y_{\mathcal{I}}} \tilde{Y}_{\mathcal{I},y}$  lifting the finite  $\tilde{Y}_{\mathcal{K},y}$ -scheme  $U_{\mathcal{K},y}$ . Note, that for any inclusion of ideals of definition  $\mathcal{J} \subset \mathcal{I}$ , there is a canonical isomorphism  $Y_{\mathcal{I}} \times_{\mathfrak{Y}} \tilde{Y}_{\mathcal{J},y} \cong \tilde{Y}_{\mathcal{I},y}$  and we let  $\tilde{\mathfrak{Y}}_y$  be the locally noetherian formal scheme  $\varinjlim_{\mathcal{I}} \tilde{Y}_{\mathcal{I},y}$ . For a fixed ideal of definition  $\mathcal{I}$ , consider the induced commutative diagram:

$$\begin{array}{ccccc}
 U'_{\mathcal{I},y} & \xrightarrow{\quad} & \tilde{Z}_{\mathcal{I},y} & \xrightarrow{\quad} & \tilde{X}_{\mathcal{I},y} \\
 \downarrow \phi_{\mathcal{I},y} & & \swarrow & \downarrow & \searrow \\
 & & Z_{\mathcal{I}} & \xrightarrow{\quad} & X_{\mathcal{I}} \\
 & & \downarrow & & \downarrow \omega_{\mathcal{I}} \\
 U_{\mathcal{I},y} & \xrightarrow{\quad} & (\tilde{\omega}_{\mathcal{I},y})_! \tilde{Z}_{\mathcal{I}} & \xrightarrow{\quad} & \tilde{Y}_{\mathcal{I},y} \\
 & & \downarrow & \swarrow & \searrow \\
 & & (\omega_{\mathcal{I}})_! Z_{\mathcal{I}} & \xrightarrow{\quad} & Y_{\mathcal{I}}
 \end{array}$$

where every square is cartesian, except for the squares at the front and back right. Note that the bottom square is cartesian by Corollary 3.6. We deduce from here that the map  $\phi_{\mathcal{I},y} : U'_{\mathcal{I},y} \rightarrow U_{\mathcal{I},y}$  is proper and Stein and so  $\mathcal{O}_{U_{\mathcal{I},y}} = (\phi_{\mathcal{I},y})_* \mathcal{O}_{U'_{\mathcal{I},y}}$ . In particular, since  $\mathcal{U}'_y = \varinjlim_{\mathcal{I}} U'_{\mathcal{I},y}$  is a locally noetherian formal finite  $\tilde{\mathfrak{X}}_y = \varinjlim_{\mathcal{I}} X_{\mathcal{I},y}$ -scheme by [EGA, I, 10.6.3], we deduce that  $u_y : \mathcal{U}'_y \rightarrow \tilde{\mathfrak{Y}}_y$  is an adic, proper morphism of locally noetherian formal schemes. By [EGA, III, 3.4.2],  $(u_y)_* \mathcal{O}_{\mathcal{U}'_y}$  is a finite  $\mathcal{O}_{\tilde{\mathfrak{Y}}_y}$ -algebra and an application of [EGA, III, 3.4.4] shows that there is a topological isomorphism  $(u_y)_* \mathcal{O}_{\mathcal{U}'_y} \cong \varinjlim_{\mathcal{I}} (u_{\mathcal{I},y})_* \mathcal{O}_{U'_{\mathcal{I},y}}$ . Thus, we define the finite  $\tilde{\mathfrak{Y}}_y$ -scheme  $\mathcal{U}_y = \mathrm{Spf}_{\tilde{\mathfrak{Y}}_y} (u_y)_* \mathcal{O}_{\mathcal{U}'_y}$ , and we conclude that it is the formal weak geometric colimit of the directed system  $\{U_{\mathcal{I},y}\}_{\mathcal{I}}$ .

Let  $\tilde{\mathfrak{Y}} = \coprod_{y \in |\mathfrak{Y}|} \tilde{\mathfrak{Y}}_y$ . Note that  $\tilde{\mathfrak{Y}} \rightarrow \mathfrak{Y}$  is adic étale and surjective, set  $\tilde{\mathfrak{Y}}^2 = \tilde{\mathfrak{Y}} \times_{\mathfrak{Y}} \tilde{\mathfrak{Y}}$ . Now take  $\tilde{\mathfrak{X}} = \tilde{\mathfrak{Y}} \times_{\mathfrak{Y}} \mathfrak{X}$ ,  $\tilde{\mathfrak{X}}^2 = \tilde{\mathfrak{Y}}^2 \times_{\mathfrak{Y}} \mathfrak{X}$ . Take  $\tilde{\omega} : \tilde{\mathfrak{X}} \rightarrow \tilde{\mathfrak{Y}}$  and  $\tilde{\omega}^2 : \tilde{\mathfrak{X}}^2 \rightarrow \tilde{\mathfrak{Y}}^2$  to be the induced maps. Further, set  $\tilde{\mathfrak{Z}} = \tilde{\mathfrak{X}} \times_{\mathfrak{X}} \mathfrak{Z}$ ,  $\tilde{\mathfrak{Z}}^2 = \tilde{\mathfrak{X}}^2 \times_{\mathfrak{X}} \mathfrak{Z}$  and we obtain an adic étale equivalence relation  $[\tilde{\mathfrak{Z}}^2 \rightrightarrows \tilde{\mathfrak{Z}}]$ . By Lemma 3.17, it is clear that  $\mathfrak{Z}$  is a uniform formal

geometric quotient of this equivalence relation. We also have that  $(\omega_{\mathcal{J}})_! Z_{\mathcal{J}}$  is the uniform geometric quotient of the equivalence relation  $[(\tilde{\omega}_{\mathcal{J}}^2)_! \tilde{Z}_{\mathcal{J}}^2 \rightrightarrows (\tilde{\omega}_{\mathcal{J}})_! \tilde{Z}_{\mathcal{J}}]$  and so one concludes that  $\omega_{\mathcal{J}} \mathfrak{Z}$  is the quotient of the equivalence relation  $[\tilde{\omega}_{\mathcal{J}}^2 \tilde{\mathfrak{Z}}^2 \xrightarrow[t]{s} \tilde{\omega}_{\mathcal{J}} \tilde{\mathfrak{Z}}]$  in the category of topologically ringed spaces. Note that since we have not yet shown that  $\tilde{\omega}_{\mathcal{J}} \tilde{\mathfrak{Z}}$  and  $\tilde{\omega}_{\mathcal{J}}^2 \tilde{\mathfrak{Z}}^2$  are locally noetherian formal schemes which are adic, locally quasi-finite, and separated over  $\mathfrak{Y}$ , we are not in the situation to apply Lemma 3.17; we will, however, slice this equivalence relation, so that we can.

For any ideal of definition  $\mathcal{I}$  of  $\mathfrak{Y}$ , we set  $\mathcal{U}_{\mathcal{J}} = \coprod_{y \in |\mathfrak{Y}|} \mathcal{U}_{\mathcal{J}, y}$ , and observe that  $\mathcal{U}_{\mathcal{J}} \rightarrow (\omega_{\mathcal{J}})_! Z_{\mathcal{J}}$  is an étale surjection. Let  $s_{\mathcal{J}}, t_{\mathcal{J}} : (\tilde{\omega}_{\mathcal{J}}^2)_! \tilde{Z}_{\mathcal{J}}^2 \rightarrow (\tilde{\omega}_{\mathcal{J}})_! \tilde{Z}_{\mathcal{J}}$  denote the two projections, then if we set  $\mathcal{R}_{\mathcal{J}} = s_{\mathcal{J}}^{-1}(\mathcal{U}_{\mathcal{J}}) \cap t_{\mathcal{J}}^{-1}(\mathcal{U}_{\mathcal{J}})$ , then  $[\mathcal{R}_{\mathcal{J}} \xrightarrow[t_{\mathcal{J}}]{s_{\mathcal{J}}} \mathcal{U}_{\mathcal{J}}]$  is an étale equivalence relation, and the scheme  $(\omega_{\mathcal{J}})_! Z_{\mathcal{J}}$  is a uniform geometric quotient of this equivalence relation. Next, we let  $\mathcal{R}'_{\mathcal{J}} \subset \tilde{Z}_{\mathcal{J}}^2$  and  $\mathcal{U}'_{\mathcal{J}} \subset \tilde{Z}_{\mathcal{J}}$  denote the preimages. In particular, the étale equivalence relation  $[\mathcal{R}'_{\mathcal{J}} \xrightarrow[t_{\mathcal{J}}]{s_{\mathcal{J}}} \mathcal{U}'_{\mathcal{J}}]$  has geometric quotient the scheme  $Z_{\mathcal{J}}$ . By Corollary 3.5, we have  $(\tilde{\omega}_{\mathcal{J}})_! \mathcal{U}'_{\mathcal{J}} = \mathcal{U}_{\mathcal{J}}$  and  $(\tilde{\omega}_{\mathcal{J}}^2)_! \mathcal{R}'_{\mathcal{J}} = \mathcal{R}_{\mathcal{J}}$ . Hence,  $(\omega_{\mathcal{J}})_! Z_{\mathcal{J}}$  is the uniform geometric quotient of the étale equivalence relation  $[(\tilde{\omega}_{\mathcal{J}}^2)_! \mathcal{R}'_{\mathcal{J}} \xrightarrow[t_{\mathcal{J}}]{s_{\mathcal{J}}} (\tilde{\omega}_{\mathcal{J}})_! \mathcal{U}'_{\mathcal{J}}]$ .

Define  $\mathcal{U} := \varinjlim_{\mathcal{J}} \mathcal{U}_{\mathcal{J}} = \coprod_{y \in |\mathfrak{Y}|} \mathcal{U}_y \subset \tilde{\omega}_{\mathcal{J}} \tilde{\mathfrak{Z}}$ , which is a clopen subspace, a locally noetherian formal scheme, that is adic, locally quasi-finite, and separated over  $\mathfrak{Y}$ . Next, take  $\mathfrak{R} := \varinjlim_{\mathcal{J}} \mathcal{R}_{\mathcal{J}} = s^{-1}(\mathcal{U}) \cap t^{-1}(\mathcal{U}) \subset \tilde{\omega}_{\mathcal{J}}^2 \tilde{\mathfrak{Z}}^2$ , then as topologically ringed spaces, we have:

$$\mathfrak{R} = \coprod_{y_s, y_t \in |\mathfrak{Y}|} s^{-1}(\mathcal{U}_{y_s}) \cap t^{-1}(\mathcal{U}_{y_t}).$$

Repeating the arguments showing that  $\mathcal{U}_y$  is a locally noetherian formal scheme, which is finite over  $\mathfrak{Y}$ , one concludes that  $s^{-1}(\mathcal{U}_y)$  (resp.  $t^{-1}(\mathcal{U}_y)$ ) is a locally noetherian formal scheme for all  $y \in |\mathfrak{Y}|$ , which is finite over  $\mathfrak{Y}^2$ . In particular, we conclude immediately that  $\mathfrak{R}$  is a locally noetherian formal scheme that is adic, locally quasi-finite, and separated over  $\mathfrak{Y}$ . Moreover, the maps  $s, t : \mathfrak{R} \rightarrow \mathcal{U}$  are adic étale, surjective and we obtain an adic étale equivalence relation  $[\mathfrak{R} \rightrightarrows \mathcal{U}]$ . By Lemma 3.17, this equivalence relation has a uniform formal geometric quotient,  $\mathfrak{V}$ , which is a locally noetherian formal scheme, adic, locally quasi-finite, and separated over  $\mathfrak{Y}$ . Note that the topological space  $|\mathfrak{V}| = |(\omega_{\mathcal{K}})_! Z_{\mathcal{K}}|$  is quasi-compact and so  $\mathfrak{V} \in \mathbf{QF}(\mathfrak{Y})$ . Further, by Lemma 2.14 we have isomorphisms of topologically ringed spaces:

$$\begin{aligned} \omega_{\mathcal{J}} \mathfrak{Z} &= \varinjlim_{\mathcal{J}} (\omega_{\mathcal{J}})_! Z_{\mathcal{J}} \cong \varinjlim_{\mathcal{J}} \varinjlim_{\mathcal{J}} \left[ (\tilde{\omega}_{\mathcal{J}}^2)_! \mathcal{R}'_{\mathcal{J}} \xrightarrow[t_{\mathcal{J}}]{s_{\mathcal{J}}} (\tilde{\omega}_{\mathcal{J}})_! \mathcal{U}'_{\mathcal{J}} \right] \\ &\cong \varinjlim_{\mathcal{J}} \left[ \varinjlim_{\mathcal{J}} (\tilde{\omega}_{\mathcal{J}}^2)_! \mathcal{R}'_{\mathcal{J}} \xrightarrow[\varinjlim_{\mathcal{J}} t_{\mathcal{J}}]{\varinjlim_{\mathcal{J}} s_{\mathcal{J}}} \varinjlim_{\mathcal{J}} (\tilde{\omega}_{\mathcal{J}})_! \mathcal{U}'_{\mathcal{J}} \right] \cong \varinjlim_{\mathcal{J}} \left[ \mathfrak{R} \xrightarrow[t]{s} \mathcal{U} \right] \cong \mathfrak{V}. \end{aligned}$$

Since the colimits appearing above were all weak formal geometric colimits, we have shown that  $\omega_{\mathcal{J}} \mathfrak{Z}$  is a locally noetherian formal scheme, and is the weak formal geometric colimit of the directed system  $\{(\omega_{\mathcal{J}})_! Z_{\mathcal{J}}\}_{\mathcal{J}}$ . By Lemma 2.14, we conclude that  $\omega_{\mathcal{J}} \mathfrak{Z}$  is the categorical colimit of its defining directed system in  $\mathbf{QF}(\mathfrak{Y})$ . Also, there is an adic, proper, surjective, and Stein map  $\omega^{\mathfrak{Z}} : \mathfrak{Z} \rightarrow \omega_{\mathcal{J}} \mathfrak{Z}$ .



We now check that  $\mathfrak{Z} \mapsto \omega_! \mathfrak{Z}$  defines a functor. So, given a morphism  $\mathfrak{Z} \xrightarrow{f} \mathfrak{Z}^0$ , there is a canonical morphism  $f_{\mathfrak{J}} : Z_{\mathfrak{J}} \rightarrow Z_{\mathfrak{J}}^0$  for any ideal of definition  $\mathfrak{J}$  of  $\mathfrak{Y}$ . Since  $(\omega_{\mathfrak{J}})_!$  is a functor, we obtain a morphism  $(\omega_{\mathfrak{J}})_! f_{\mathfrak{J}} : (\omega_{\mathfrak{J}})_! Z_{\mathfrak{J}} \rightarrow (\omega_{\mathfrak{J}})_! Z_{\mathfrak{J}}^0$ . Taking the categorical colimit of this family of morphisms in  $\mathbf{QF}(\mathfrak{Y})$  produces a morphism of formal schemes  $\omega_! f : \omega_! \mathfrak{Z} \rightarrow \omega_! \mathfrak{Z}^0$ . That this construction respects composition is immediate from the functoriality of  $(\omega_{\mathfrak{J}})_!$ .

To show that  $(\omega_!, \omega^*)$  is an adjoint pair, let  $\mathfrak{Z} \in \mathbf{QF}_{p.f.}(\mathfrak{X}/\mathfrak{Y})$  and  $\mathfrak{Z}^1 \in \mathbf{QF}(\mathfrak{Y})$ , then

$$\begin{aligned} \mathrm{Hom}_{\mathbf{QF}(\mathfrak{Y})}(\omega_! \mathfrak{Z}, \mathfrak{Z}^1) &= \mathrm{Hom}_{\mathbf{QF}(\mathfrak{Y})}(\varinjlim_{\mathfrak{J}} (\omega_{\mathfrak{J}})_! Z_{\mathfrak{J}}, \mathfrak{Z}^1) = \varprojlim_{\mathfrak{J}} \mathrm{Hom}_{\mathbf{QF}(\mathfrak{Y})}((\omega_{\mathfrak{J}})_! Z_{\mathfrak{J}}, \mathfrak{Z}^1) \\ &= \varprojlim_{\mathfrak{J}} \mathrm{Hom}_{\mathbf{QF}(\mathfrak{Y}_{\mathfrak{J}})}((\omega_{\mathfrak{J}})_! Z_{\mathfrak{J}}, Z_{\mathfrak{J}}^1) = \varprojlim_{\mathfrak{J}} \mathrm{Hom}_{\mathbf{QF}_{p.f.}(\mathfrak{X}_{\mathfrak{J}}/\mathfrak{Y}_{\mathfrak{J}})}(Z_{\mathfrak{J}}, (\omega_{\mathfrak{J}})^* Z_{\mathfrak{J}}^1) \\ &= \varprojlim_{\mathfrak{J}} \mathrm{Hom}_{\mathbf{QF}_{p.f.}(\mathfrak{X}/\mathfrak{Y})}(Z_{\mathfrak{J}}, \omega^* \mathfrak{Z}^1) = \mathrm{Hom}_{\mathbf{QF}_{p.f.}(\mathfrak{X}/\mathfrak{Y})}(\mathfrak{Z}, \omega^* \mathfrak{Z}^1). \end{aligned}$$

The remainder of the claims are easily verified, so they are omitted.  $\square$

Before we prove Corollary 3.15, we require a basic result on adic, proper, and Stein morphisms of locally noetherian formal schemes.

**Lemma 3.18.** *Let  $\omega : \mathfrak{X} \rightarrow \mathfrak{Y}$  be a proper, adic, morphism of locally noetherian formal schemes. Given an adic flat map  $p : \mathfrak{Y}' \rightarrow \mathfrak{Y}$  of locally noetherian formal schemes, let  $\omega' : \mathfrak{X}' \rightarrow \mathfrak{Y}'$  be the pullback of  $\omega$  by  $p$ . Let  $\mathfrak{F}$  be a coherent  $\mathcal{O}_{\mathfrak{X}}$ -module. Then the base change homomorphism  $p^* \omega_* \mathfrak{F} \rightarrow \omega'_* p'^* \mathfrak{F}$  is a topological isomorphism. In particular, proper, adic, Stein morphisms are preserved under adic flat base change.*

*Proof.* Note that this result is well-known for the case of a morphism of noetherian schemes. For the formal case, we are unaware of a reference. By [EGA, III, 3.4.5.1], the statement is Zariski local on  $\mathfrak{Y}$  and  $\mathfrak{Y}'$ , so we may assume that  $\mathfrak{Y} = \mathrm{Spf}_I R$  for some  $I$ -adic noetherian ring  $R$ , and  $\mathfrak{Y}' = \mathrm{Spf}_{I R'} R'$  and  $R \rightarrow R'$  is an adic flat morphism of adic noetherian rings. In particular, we observe that  $R \rightarrow R'$  is a flat morphism of abstract rings. We need to show that the morphism  $\Gamma(\mathfrak{X}, \mathfrak{F}) \widehat{\otimes}_R R' \rightarrow \Gamma(\mathfrak{X}', p'^* \mathfrak{F})$  is a topological isomorphism. By [EGA, III, 3.4.2], the modules  $\Gamma(\mathfrak{X}, \mathfrak{F})$ ,  $\Gamma(\mathfrak{X}', p'^* \mathfrak{F})$  are coherent  $R'$ -modules, thus to show that the map is a topological isomorphism, by [EGA, I, 10.11.6], it suffices to show it is an abstract isomorphism of  $R'$ -modules. This is covered by [AJL99, Prop. I.7.2].  $\square$

**Lemma 3.19.** *Consider a commutative diagram of locally noetherian formal schemes:*

$$\begin{array}{ccc} & \mathfrak{X} & \\ f \swarrow & & \searrow g \\ \mathfrak{Y} & \xrightarrow{h} & \mathfrak{Z} \end{array}$$

*where  $f, g$  are adic, proper, surjective, and have geometrically connected fibers. If  $h$  is quasi-finite, then it is a universal homeomorphism. If, in addition,  $f, g$  are Stein, then  $h$  is an isomorphism of formal schemes.*

*Proof.* By restricting to the underlying reduced closed subschemes, the first claim follows from Lemma 3.8. The latter claim is obvious.  $\square$

*Proof of Corollary 3.15.* We note that (3) is clear, and concentrate on (1) and (2). As in the proof of Corollary 3.6, for  $(\mathfrak{Z} \xrightarrow{\sigma} \mathfrak{X}) \in \mathbf{QF}_{\text{p.f.}}(\mathfrak{X}/\mathfrak{Y})$  we obtain a commutative diagram:

$$\begin{array}{ccccc}
 & & \mathfrak{p}^* \mathfrak{Z} & \xrightarrow{\quad} & \mathfrak{Z} \\
 & \swarrow & \downarrow & \searrow & \downarrow \\
 & & \mathfrak{X}' & \xrightarrow{\quad p' \quad} & \mathfrak{X} \\
 & \swarrow & \downarrow & \searrow & \downarrow \\
 \omega'_! \mathfrak{p}^* \mathfrak{Z} & \xrightarrow{\quad} & \mathfrak{p}^* \omega'_! \mathfrak{Z} & \xrightarrow{\quad} & \omega'_! \mathfrak{Z} \\
 & \searrow & \downarrow & \searrow & \downarrow \\
 & & \mathfrak{Y}' & \xrightarrow{\quad p \quad} & \mathfrak{Y}
 \end{array}$$

We note that the base change map  $\omega'_! \mathfrak{p}^* \mathfrak{Z} \rightarrow \mathfrak{p}^* \omega'_! \mathfrak{Z}$  is quasi-finite. Moreover, by Theorem 3.12(1b), the map  $\mathfrak{p}^* \mathfrak{Z} \rightarrow \omega'_! \mathfrak{p}^* \mathfrak{Z}$  is adic, proper, Stein, surjective, and has geometrically connected fibers. Similarly, we conclude that the map  $\mathfrak{p}^* \mathfrak{Z} \rightarrow \mathfrak{p}^* \omega'_! \mathfrak{Z}$  is adic, proper, surjective and has geometrically connected fibers, and if  $p$  is flat the morphism is also Stein by Lemma 3.18. By Lemma 3.19 the base change map  $\Delta : \omega'_! \mathfrak{p}^* \mathfrak{Z} \rightarrow \mathfrak{p}^* \omega'_! \mathfrak{Z}$  is a finite, universal homeomorphism and is an isomorphism if  $p$  is flat.  $\square$

**3.3. The Theorem on Formal Functions.** Let  $\pi : X \rightarrow Y$  be a proper morphism of locally noetherian schemes, let  $|V| \subset |Y|$  be a closed subset, then there is a 2-commutative diagram of categories:

$$\begin{array}{ccc}
 \mathbf{QF}(Y) & \xrightarrow{c_{/|V|}} & \mathbf{QF}(\widehat{Y}_{/|V|}) \\
 \pi^* \downarrow & & \downarrow \widehat{\pi}^* \\
 \mathbf{QF}(X) & \xrightarrow{c_{/\pi^{-1}|V|}} & \mathbf{QF}(\widehat{X}_{/\pi^{-1}|V|})
 \end{array}$$

From this, we obtain a natural transformation of functors from  $\mathbf{QF}_{\text{p.f.}}(X/Y)$  to  $\mathbf{QF}(\widehat{Y})$ :

$$\Phi_{\pi, |V|} : \widehat{\pi}_! \circ c_{/\pi^{-1}|V|} \implies c_{/|V|} \circ \pi_!.$$

**Theorem 3.20 (Formal Functions).** *Suppose that  $\pi : X \rightarrow Y$  is a proper morphism of locally noetherian schemes and let  $|V| \subset |Y|$  be a closed subset, then  $\Phi_{\pi, |V|}$  is an isomorphism of functors. That is, for any  $Z$  in  $\mathbf{QF}_{\text{p.f.}}(X/Y)$ , there is a natural isomorphism*

$$\widehat{\pi}_! \widehat{Z} \longrightarrow (\pi_! Z)^\wedge.$$

*Proof.* Let  $\mathcal{K} \triangleleft \mathcal{O}_Y$  denote the unique radical ideal with support  $|V|$ ; this is the largest ideal of definition for the locally noetherian formal scheme  $\widehat{Y}_{/|V|}$  and any other ideal of definition for  $\widehat{Y}_{/|V|}$  is given by an ideal  $\mathcal{J} \subset \mathcal{K}$  with support  $|V|$ . For any ideal of definition  $\mathcal{J}$ , we set  $Y_{\mathcal{J}} = V(\mathcal{J})$ ,  $X_{\mathcal{J}} = X \times_Y Y_{\mathcal{J}}$ , and denote the induced map as  $\pi_{\mathcal{J}} : X_{\mathcal{J}} \rightarrow Y_{\mathcal{J}}$ , also let  $Z_{\mathcal{J}} = Z \times_X X_{\mathcal{J}}$ . By Corollary 3.6(1), there is a natural, finite, universal homeomorphism  $\jmath_{\mathcal{J}} : (\pi_{\mathcal{K}})_! Z_{\mathcal{K}} \rightarrow (\pi_{\mathcal{J}})_! Z_{\mathcal{J}}$ . By the construction in Theorem 3.12, we see that  $\widehat{\pi}_! \widehat{Z}$  is the topologically locally ringed space  $(|(\pi_{\mathcal{K}})_! Z_{\mathcal{K}}|, \varprojlim_{\mathcal{J}} \jmath_{\mathcal{J}}^{-1} \mathcal{O}_{(\pi_{\mathcal{J}})_! Z_{\mathcal{J}}})$ . Also,  $(\pi_! Z)^\wedge$  is defined as the topologically locally ringed space  $(|(\pi_! Z)_{\mathcal{K}}|, \widehat{\mathcal{O}}_{\pi_! Z})$ . The map  $\Phi : \widehat{\pi}_! \widehat{Z} \rightarrow (\pi_! Z)^\wedge$  is on the level of topological spaces  $|\Phi| : |(\pi_{\mathcal{K}})_! Z_{\mathcal{K}}| \rightarrow |(\pi_! Z)_{\mathcal{K}}|$ , which is a finite, universal homeomorphism by Corollary 3.6(1). On the level of sheaves of topological rings, the

map is  $\Phi^\# : \widehat{\mathcal{O}}_{\pi_! Z} \rightarrow \Phi_* \varprojlim_j \mathcal{O}_{(\pi_j)_! Z_j}^{-1}$ . By Theorem 3.3(1b), we have natural, proper and Stein morphisms  $\pi^Z : Z \rightarrow \pi_! Z$  and  $\pi_j^{Z_j} : Z_j \rightarrow (\pi_j)_! Z_j$  so there are isomorphisms of sheaves  $\mathcal{O}_{\pi_! Z} = (\pi^Z)_* \mathcal{O}_Z$  and  $\mathcal{O}_{(\pi_j)_! Z_j} = (\pi_j^{Z_j})_* \mathcal{O}_{Z_j}$ . Thus, the map on sheaves of topological rings is seen to be

$$(\pi_*^Z \mathcal{O}_Z)^\wedge \longrightarrow \Phi_* \varprojlim_j \mathcal{O}_{(\pi_j)_! Z_j}^{-1} (\pi_j^{Z_j})_* \mathcal{O}_{Z_j},$$

which is a topological isomorphism by [EGA, III, 3.4.4].  $\square$

#### 4. THE EXISTENCE THEOREM

**Definition 4.1.** For a morphism of algebraic stacks  $\pi : X \rightarrow Y$ , let  $\mathbf{QF}_p(X/Y)$  denote the full subcategory of  $\mathbf{QF}_{p,f.}(X/Y)$  consisting of those  $(Z \rightarrow X)$  such that the composition  $Z \rightarrow X \rightarrow Y$  is proper.

Let  $R$  be an  $I$ -adic noetherian ring, set  $S = \operatorname{Spec} R$ ,  $S_n = \operatorname{Spec} R/I^{n+1}$ . Further, suppose that  $\pi : X \rightarrow S$  is a morphism of locally noetherian algebraic stacks. We set  $X_n = X \times_S S_n$  and let  $\pi_n : X_n \rightarrow S_n$  denote the induced morphism. Consider the  $I$ -adic completion functor

$$\Psi_{\pi,I} : \mathbf{QF}_p(X/S) \longrightarrow \varprojlim_n \mathbf{QF}_p(X_n/S_n) : (Z \rightarrow X) \mapsto (Z \times_S S_n \rightarrow X_n)_{n \geq 0},$$

where the category  $\varprojlim_n \mathbf{QF}_p(X_n/S_n)$  is the 2-categorical limit of the inverse system of categories  $\{\mathbf{QF}_p(X_n/S_n)\}_{n \geq 0}$ . The purpose of an existence theorem is to provide sufficient conditions on the morphism  $\pi : X \rightarrow S$  for the functor  $\Psi_{\pi,I}$  to be an equivalence of categories.

In the case that the morphism  $X \rightarrow S$  is locally of finite type and *separated*, the category  $\mathbf{QF}_p(X/S)$  is equivalent to the category of finite algebras over  $X$  with  $S$ -proper support. A consequence of [Ols05, Thm. 1.4] is that the functor  $\Psi_{\pi,I}$  is an equivalence. In full generality, we have the following

**Lemma 4.2.** *Let  $S$  be the spectrum of an  $I$ -adic noetherian ring  $R$ . Consider a locally of finite type morphism of algebraic stacks  $\pi : X \rightarrow S$ , then the functor  $\Psi_{\pi,I}$  is fully faithful.*

*Proof.* Consider objects  $Z, Z' \in \mathbf{QF}_p(X/S)$  and an adic system of morphisms  $(f_n)_{n \geq 0} : (Z_n)_{n \geq 0} \rightarrow (Z'_n)_{n \geq 0}$ . In particular,  $(f_n)_{n \geq 0}$  is an adic system of finite morphisms to a proper  $S$ -stack, so by [Ols05, Thm. 1.4], there is a finite  $S$ -morphism  $f : Z \rightarrow Z'$  such that for every  $n \geq 0$ , there is a 2-isomorphism  $f_{S_n} \cong f_n$ . We must now show that  $f$  is automatically an  $X$ -morphism, which will follow if we can prove that given two  $S$ -morphisms  $s, t : Z \rightarrow X$  such that for every  $n \geq 0$  we have a 2-isomorphism  $s_{X_n} \cong t_{X_n}$ , then  $s \cong t$ . One works smooth locally and considers the induced morphisms of formal schemes, to apply [EGA, I, 10.9.4] and concludes that  $s = t$  on an open subset of the stack  $Z$ , which contains the closed fiber. Since the stack  $Z$  is  $S$ -proper and  $S$  is  $I$ -adic, we conclude immediately that  $s = t$  on the stack  $Z$ . Thus, we have shown that the functor  $\Psi_{\pi,I}$  is full. To obtain the faithfulness, we suppose that we have two finite  $X$ -morphisms  $f, g : Z \rightarrow Z'$  such that we have an equality of morphisms  $(\Psi_{\pi,I})_*(f) = (\Psi_{\pi,I})_*(g)$  in the category  $\varprojlim_n \mathbf{QF}_p(X_n/S_n)$ , then we have to show that  $f = g$  in  $\mathbf{QF}_p(X/S)$ . Once again, we apply [EGA, I, 10.9.4] and conclude as before. Thus, the functor  $\Psi_{\pi,I}$  is also faithful.  $\square$

**Definition 4.3.** For a morphism of algebraic stacks  $\pi : X \rightarrow Y$ , define the category  $\mathbf{QF}_{p,\text{fin } \Delta}(X/Y)$  to be the full subcategory of  $\mathbf{QF}_p(X/Y)$  consisting of those objects  $(Z \rightarrow X)$  such that the composition of morphisms  $Z \rightarrow X \rightarrow Y$  has finite diagonal.

*Remark 4.4.* In the case that the morphism of algebraic stacks  $\pi : X \rightarrow Y$  has affine stabilizers, then we have an *equality* of categories  $\mathbf{QF}_{p, \text{fin } \Delta}(X/Y) = \mathbf{QF}_p(X/Y)$ —this holds in particular for schemes, algebraic spaces and stacks with quasi-finite diagonal.

Retaining the notation used above, there is an induced completion functor:

$$\tilde{\Psi}_{\pi, I} : \mathbf{QF}_{p, \text{fin } \Delta}(X/S) \longrightarrow \varprojlim_n \mathbf{QF}_{p, \text{fin } \Delta}(X_n/S_n).$$

What follows is the main result of this section.

**Theorem 4.5.** *Let  $R$  be an  $I$ -adic noetherian ring, set  $S = \text{Spec } R$ . If the morphism of algebraic stacks  $\pi : X \rightarrow S$  is locally of finite type with quasi-compact and separated diagonal, then the functor  $\tilde{\Psi}_{\pi, I}$  is an equivalence.*

The new content is the removal of the separation hypothesis. In particular, this recovers the consequences of [Ols05, Thm. 1.4] for all locally of finite type morphisms of algebraic stacks  $\pi : X \rightarrow S$  with affine stabilizers. Some interesting special cases of this are:

- (1) global quotient stacks;
- (2) stacks with quasi-finite diagonal;
- (3) algebraic spaces;
- (4) schemes.

We will prove Theorem 4.5 in the case of a finite type morphism of schemes using a method of dévissage twice—once to handle the case of schemes, and the second to handle algebraic stacks. These methods will require a number of lemmata to set up. We start with a variant of the Artin-Rees lemma.

**Lemma 4.6.** *Let  $\mathfrak{Y}$  be a noetherian formal scheme. Consider an inclusion of coherent  $\mathcal{O}_{\mathfrak{Y}}$ -modules  $\mathfrak{F}' \subset \mathfrak{F}$  such that for a coherent ideal  $\mathfrak{J} \triangleleft \mathcal{O}_{\mathfrak{Y}}$  we have  $\mathfrak{J}\mathfrak{F}' = (0)$ . Then there is a  $k \geq 0$  such that  $(\mathfrak{J}^k \mathfrak{F}) \cap \mathfrak{F}' = (0)$ .*

*Proof.* Since  $\mathfrak{Y}$  is quasicompact, this is Zariski local on  $\mathfrak{Y}$ . Then it is simply a matter of applying the Artin-Rees Lemma [AM69, Cor. 10.10].  $\square$

Let  $f : \mathfrak{Z}' \rightarrow \mathfrak{Z}$  be a finite morphism of locally noetherian formal schemes. We denote the induced map on sheaves of rings by  $f^\# : \mathcal{O}_{\mathfrak{Z}} \rightarrow f_* \mathcal{O}_{\mathfrak{Z}'}$ . As is customary, we will define the **conductor ideal** of  $f$  to be  $\mathfrak{C}_f := \text{Ann}_{\mathcal{O}_{\mathfrak{Z}}}(f_* \mathcal{O}_{\mathfrak{Z}'}/\mathcal{O}_{\mathfrak{Z}})$ .

**Lemma 4.7.** *Suppose that  $X$  is a noetherian scheme,  $|V| \subset |X|$  a closed subset. Consider a commutative diagram of noetherian formal schemes:*

$$\begin{array}{ccc} \mathfrak{Z}' & \xrightarrow{f} & \mathfrak{Z} \\ & \searrow s' & \swarrow s \\ & \widehat{X}_{|V|} & \end{array}$$

where  $s, s'$  are adic, quasi-finite, and separated;  $f$  is finite, surjective, and  $\mathfrak{Z}'$  is effectivizable. If

- (1) *there is a coherent ideal  $\mathfrak{J} \triangleleft \mathcal{O}_{\mathfrak{Z}}$  such that  $\mathfrak{J} \cap \ker f^\# = (0)$  and  $V(\mathfrak{J}), V(\mathfrak{J}_{\mathfrak{Z}'})$  are effectivizable, then there is a unique factorization of  $f : \mathfrak{Z}' \rightarrow \mathfrak{Z}'' \xrightarrow{\beta} \mathfrak{Z}$  such that  $\beta$  is finite, surjective,  $\beta^\# : \mathcal{O}_{\mathfrak{Z}} \rightarrow \beta_* \mathcal{O}_{\mathfrak{Z}''}$  is injective,  $\mathfrak{Z}''$  is effectivizable, the formation of  $\mathfrak{Z}'' \xrightarrow{\beta} \mathfrak{Z}$  commutes with flat base change on  $X$ , and  $\mathfrak{C}_f \subset \mathfrak{C}_\beta$ ; or*
- (2) *there is a coherent  $\mathcal{O}_X$ -ideal  $\mathfrak{J}$  such that  $\ker f^\#$  and  $\text{coker } f^\#$  are annihilated by  $\mathfrak{J}_{\mathfrak{Z}}$ , and  $V(\mathfrak{J}_{\mathfrak{Z}}^n)$  is effectivizable for all  $n \geq 1$ . Then  $\mathfrak{Z}$  is effectivizable.*

*Proof.* For (1), we define  $\mathfrak{Z}''$  to be the colimit of the diagram  $[V(\mathcal{J}) \leftarrow V(\mathcal{J}_{\mathfrak{Z}'}) \hookrightarrow \mathfrak{Z}']$  in  $\mathbf{QF}(\widehat{X}_{/|V|})$ , which exists and is effectivizable by Theorem 2.18 and Corollary 2.19. There is an induced factorization  $\mathfrak{Z}' \rightarrow \mathfrak{Z}'' \xrightarrow{\beta} \mathfrak{Z}$  which commutes with flat base change on  $X$ . It remains to show that the map  $\beta^\#$  is injective. This is a local problem, so we may assume that  $\mathfrak{Z} = \mathrm{Spf}_I A$  and  $\mathfrak{Z}' = \mathrm{Spf}_{IB} B$  for some finite map  $G : A \rightarrow B$  of noetherian rings, where  $B$  is  $IB$ -adic. We write  $J = \Gamma(\mathrm{Spf}_I A, \mathcal{J})$ ,  $K = \ker G$ , and Theorem 2.18 implies that  $\mathfrak{Z}'' = \mathrm{Spf}_{I'}(A/J \times_{B/BJ} B)$ , where  $I'$  is the ideal generated by  $I$  in  $A/J \times_{B/BJ} B$ . Thus, we have to show that the finite map of rings  $b : A \rightarrow A/J \times_{B/BJ} B$  is injective, which is obvious, since  $J \cap K = (0)$ . It is clear that  $\mathrm{coker} b = BJ/J$  and so  $\mathcal{C}_G \subset \mathcal{C}_b$ .

For (2), by Lemma 4.6, we may replace  $\mathcal{J}_3$  by some power such that  $\ker f^\# \cap \mathcal{J}_3 = (0)$ . By (1), there is a factorization of  $f : \mathfrak{Z}' \rightarrow \mathfrak{Z}$  as  $\mathfrak{Z}' \rightarrow \mathfrak{Z}'' \xrightarrow{\beta} \mathfrak{Z}$ , where  $\mathfrak{Z}''$  is effectivizable, the map of formal schemes  $\beta$  is finite and surjective,  $\beta^\#$  injective, and  $\mathrm{coker} \beta^\#$  is annihilated by  $\mathcal{J}_3$ . In particular, since  $\mathcal{J}_3 \subset \mathcal{C}_\beta$ , then there is a cocartesian diagram of formal  $\widehat{X}$ -schemes:

$$\begin{array}{ccc} \widehat{V(\mathcal{J}_{\mathfrak{Z}''})}_{/|V|} & \hookrightarrow & \mathfrak{Z}'' \\ \downarrow & & \downarrow \beta \\ V(\mathcal{J}_3) & \hookrightarrow & \mathfrak{Z}. \end{array}$$

By Corollary 2.19, we conclude that  $\mathfrak{Z}$  is effectivizable.  $\square$

We have a simple lemma that will be used shortly.

**Lemma 4.8.** *Let  $R$  be an  $I$ -adically complete noetherian ring. Define  $V = \mathrm{Spec} R$  and  $\widehat{V} = \mathrm{Spf}_I R$ . Let  $\mathfrak{Z} \rightarrow \widehat{V}$  be a quasi-compact and adic morphism. Fix a coherent sheaf  $\mathfrak{F}$  on  $\mathfrak{Z}$ . Let  $\mathfrak{p}$  be an open prime of  $R$ , and let  $R'$  be the  $\mathfrak{p}$ -adic completion of  $R_{\mathfrak{p}}$ . Take  $\widehat{V}' = \mathrm{Spf}_{IR'} R'$ . Suppose that  $\mathfrak{F}_{\widehat{V}'} = 0$  on  $\widehat{\mathfrak{Z}}_{\widehat{V}'}$ , then there is an open neighborhood  $\mathcal{U}$  of  $v = [\mathfrak{p}] \in \widehat{V}$  such that  $\mathfrak{F}_{\mathcal{U}} = 0$  on  $\mathfrak{Z}_{\mathcal{U}}$ .*

*Proof.* Let  $V_0 = \mathrm{Spec}(R/I)$ ,  $V'_0 = \mathrm{Spec}(R'/IR')$ ,  $Z_0 = \mathfrak{Z} \times_{\widehat{V}} V_0$ , and  $F_0 = \mathfrak{F}|_{Z_0}$ . Next, observe that  $R'/IR'$  is the completion of  $R_{\mathfrak{p}}/IR_{\mathfrak{p}}$  with respect to the  $\mathfrak{p}$ -adic topology and thus  $R_{\mathfrak{p}}/IR_{\mathfrak{p}} \rightarrow R'/IR'$  is faithfully flat. Hence, if  $W_0 = \mathrm{Spec}(R_{\mathfrak{p}}/IR_{\mathfrak{p}})$ , then since  $F_0|_{(Z_0)_{V'_0}} = 0$ , we have  $F_0|_{(Z_0)_{W_0}} = 0$ . In particular, it follows from the result for schemes, that there is an open neighborhood  $\mathcal{U}_0$  of  $v$  in  $V_0$  such that  $F_0|_{(Z_0)_{\mathcal{U}_0}} = 0$ . One now obtains from  $\mathcal{U}_0$  a formal open subscheme  $\mathcal{U}$  of  $\widehat{V}$  such that of  $\mathfrak{F}|_{\mathcal{U}}$  to  $(Z_0)_{\mathcal{U}_0}$  is 0. Since  $\mathfrak{Z}_{\mathcal{U}}$  is a noetherian formal scheme and  $\mathfrak{F}|_{\mathcal{U}}$  is coherent, then  $\mathfrak{F}|_{\mathcal{U}} = 0$  on  $\mathfrak{Z}_{\mathcal{U}}$ .  $\square$

This next result is the analogue of [EGA, III, 5.3.4] to our situation, and addresses the existence of an effectivizable formal scheme  $\mathfrak{Z}'$  which would be used in Lemma 4.7.

**Lemma 4.9.** *Let  $Y$  be a noetherian scheme,  $t : Y' \rightarrow Y$  a proper morphism, and let  $U \subset Y$  be an open subset such that  $t^{-1}U \rightarrow U$  is an isomorphism. Let  $\mathcal{J} \triangleleft \mathcal{O}_Y$  be such that  $\mathrm{supp}(\mathcal{O}_Y/\mathcal{J})^c = U$ . Further, suppose that  $|V| \subset |Y|$  is a closed subset, and let  $\widehat{t} : \widehat{Y}' \rightarrow \widehat{Y}$  denote the induced map on completions along  $|V|$ . If  $(\mathcal{E} \xrightarrow{\gamma} \widehat{Y}) \in \mathbf{QF}(\widehat{Y})$ , let  $\eta : \widehat{t}_! \widehat{t}^* \mathcal{E} \rightarrow \mathcal{E}$  denote the adjunction morphism, then there is an  $n > 0$  such that the kernel and cokernel of the map  $\mathcal{O}_{\mathcal{E}} \rightarrow \eta_* \mathcal{O}_{\widehat{t}_! \widehat{t}^* \mathcal{E}}$  is annihilated by the  $\mathcal{O}_{\mathcal{E}}$ -ideal  $\widehat{\mathcal{J}}_{\mathcal{E}}^k$ .*

*Proof.* Since  $Y$  is quasi-compact, the result is local on  $Y$  for the Zariski topology. So, we may assume that  $Y = \mathrm{Spec} R$ ,  $\mathcal{J} = \widetilde{J}$ , for  $J \triangleleft R$  and let  $I$  be an ideal defining  $|V| \subset |Y|$ .

Let  $\widehat{R}$  denote the  $I$ -adic completion of  $R$ , then we are free to replace  $R$  by  $\widehat{R}$  and we consequently assume that  $R$  is  $I$ -adic. Since  $R$  is  $I$ -adic, then any open neighbourhood of  $|V| = |\mathrm{Spec}(R/I)|$  is all of  $\mathrm{Spec} R$ . Thus, it suffices to exhibit for every  $y \in |V|$ , an open neighbourhood  $Y_y$  of  $y$  such that the result is true on the  $Y_y$ . By Lemma 4.8 combined with 3.15, we may thus assume that  $R$  is local and maximal-adically complete.

We now proceed to prove the result by induction on  $\dim R = n$ . In the case that  $n = 0$ , then  $R$  is local artinian, and the result is trivial. Otherwise, assume that  $n > 0$ , and we have proven the result for all complete local rings  $S$  with  $\dim S < n$ . Note, that by what we have proven in the previous paragraph, the inductive hypothesis ensures that we have proven the result for all noetherian schemes  $Y$  such that all the local rings of  $Y$  have  $\dim < n$ . Now, since  $R$  is maximal-adically complete, then  $R/I^n$  is maximal-adically complete for any  $n > 0$  and so if  $E_n = \mathcal{E} \times_{\mathrm{Spf}_I R} \mathrm{Spec}(R/I^{n+1})$ , by [EGA, IV, 18.12.3], there is a decomposition  $E_n = V_n^1 \amalg V_n^2$ , where  $V_n^1 \rightarrow E_n$  is finite, contains the special fiber, and  $V_n^2$  is disjoint from the special fiber. These decompositions are compatible, in the sense that  $V_n^i \times_{E_n} E_{n-1} = V_{n-1}^i$  for  $i = 1, 2$  and all  $n > 1$  and so we find that there is a decomposition of the formal scheme  $\mathcal{E} = \mathfrak{V}^1 \amalg \mathfrak{V}^2$  such that  $\mathfrak{V}^1$  is finite over  $\mathrm{Spf}_I R$  and  $\mathfrak{V}^2$  misses the special fiber.

Combining Corollary 3.16 with [EGA, III, 5.3.4] applied to  $\mathfrak{V}^1 \rightarrow \mathrm{Spec} R$  we see that there is a  $k$  such that  $\widehat{\mathcal{J}}_{\mathfrak{V}^1}^k$  annihilates the kernel and cokernel of the map  $\mathcal{O}_{\mathfrak{V}^1} \rightarrow \eta_* \mathcal{O}_{\widehat{\mathfrak{t}}_1 \widehat{\mathfrak{t}}^* \mathfrak{V}^1}$ . Now note that  $\mathfrak{V}^2 \rightarrow W := (\mathrm{Spec} R) - \{m\}$ , where  $m$  is the maximal ideal of  $R$ , and since  $W$  is a noetherian scheme with all local rings of dimension  $< n$ , applying Corollaries 3.14 and 3.15 to  $\mathfrak{V}^2 \rightarrow W$  we may use the inductive hypothesis to obtain that  $\widehat{\mathcal{J}}_{\mathfrak{V}^2}^k$  annihilates the kernel and cokernel of this map (by possibly increasing  $k$ ). Thus, since  $\mathcal{O}_{\mathcal{E}} = \mathcal{O}_{\mathfrak{V}^1} \times \mathcal{O}_{\mathfrak{V}^2}$ , we find that there is an  $k$  such that  $\widehat{\mathcal{J}}_{\mathcal{E}}^k$  annihilates the kernel and cokernel of the morphism  $\mathcal{O}_{\mathcal{E}} \rightarrow \eta_* \mathcal{O}_{\widehat{\mathfrak{t}}_1 \widehat{\mathfrak{t}}^* \mathcal{E}}$ .  $\square$

Let  $\pi : X \rightarrow S$  be a locally of finite type morphism of locally noetherian schemes, let  $\widehat{\pi}_I : \widehat{X}_I \rightarrow \widehat{S}$  denote the  $I$ -adic completion. In particular, we define  $\mathbf{QF}_p(\widehat{X}_I/\widehat{S})$  as the full subcategory of  $\mathbf{QF}(\widehat{X}_I)$  consisting of those  $(\mathfrak{Z} \xrightarrow{\sigma} \widehat{X}_I)$  such that  $\widehat{\pi}_I \circ \sigma$  is adic proper, then we obtain an equivalence of categories:

$$\mathbf{QF}_p(\widehat{X}_I/\widehat{S}) \rightarrow \varprojlim_n \mathbf{QF}_p(X_n/S_n).$$

By abuse of notation, we will consider,  $\Psi_{\pi, I}$  to be a functor with target category  $\mathbf{QF}_p(\widehat{X}_I/\widehat{S})$ . It will sometimes be convenient to abuse notation, and suppress the subscripted  $I$  in the definition of  $\widehat{X}_I$ .

**Proposition 4.10** (Birational dévissage). *Let  $\pi : X \rightarrow S$  be a morphism of noetherian schemes, such that for any closed subscheme  $V \hookrightarrow X$ , there is a proper, birational (on  $V$ ) morphism  $\tau_V : V' \rightarrow V$  for which  $\Psi_{\pi \circ \tau_V, I}$  is an equivalence, then  $\Psi_{\pi, I}$  is an equivalence.*

*Proof.* First, let  $\iota_V : V \hookrightarrow X$  be a closed subscheme, then there is a commutative diagram of categories:

$$\begin{array}{ccc} \mathbf{QF}_p(V/S) & \xrightarrow{\Psi_{\iota_V \circ \pi, I}} & \mathbf{QF}_p(\widehat{V}/\widehat{S}) \\ (\iota_V)! \downarrow & & \downarrow (\widehat{\iota_V})_! \\ \mathbf{QF}_p(X/S) & \xrightarrow{\Psi_{\pi, I}} & \mathbf{QF}_p(\widehat{X}/\widehat{S}) \end{array}$$

The vertical maps are fully faithful inclusions of subcategories. We observe that the essential image of the functor  $(\widehat{\iota_V})_!$  is described by those objects  $(\mathfrak{Z} \xrightarrow{\sigma} \widehat{X}) \in \mathbf{QF}_p(\widehat{X}/\widehat{S})$  such that the natural map  $(\widehat{\iota_V})_!(\widehat{\iota_V})^* \mathfrak{Z} \rightarrow \mathfrak{Z}$  is an isomorphism. Similarly for the essential image of  $\iota_V$ .

By Lemma 4.2, it suffices to prove that the functor  $\Psi_{\pi, I}$  is essentially surjective, which we will show by noetherian induction on the closed subsets of  $X$ . More precisely, for a closed subset  $|V| \hookrightarrow |X|$  we will consider the statement  $P(|V|)$ : for any closed subscheme  $\iota_W : W \hookrightarrow X$  with  $|W| \subset |V|$ , then the functor  $\Psi_{\pi \circ \iota_W, I}$  is essentially surjective. Since  $P(\emptyset)$  is trivially true, by the principle of noetherian induction, the result will follow if we can show that if the statement  $P(|V'|)$  is true for all proper closed subsets  $|V'| \subsetneq |V|$ , then the statement  $P(|V|)$  is true. The statement of the lemma allows us to equivalently prove the following: if the functor  $\Psi_{\pi \circ \iota_V, I}$  is essentially surjective for any closed subscheme  $\iota_V : V \hookrightarrow X$  such that  $|V| \subsetneq |X|$ , then the functor  $\Psi_{\pi, I}$  is an equivalence.

We let  $\mathfrak{Z} := (\mathfrak{Z} \xrightarrow{\sigma} \widehat{X})$  be an object of  $\mathbf{QF}_p(\widehat{X}/\widehat{S})$  such that for any closed subscheme  $\iota_V : V \hookrightarrow X$ , with  $|V| \subsetneq |X|$ , the map  $(\widehat{\iota_V})_!(\widehat{\iota_V})^* \mathfrak{Z} \rightarrow \mathfrak{Z}$  is not an isomorphism, and by noetherian induction, it remains to show that there is a  $Z$  in  $\mathbf{QF}_p(X/S)$  such that  $\widehat{Z} \cong \mathfrak{Z}$ .

By assumption, there is a proper and birational morphism of schemes  $\tau : X' \rightarrow X$  which is an isomorphism over an open subscheme  $\mathfrak{y} : \mathcal{U} \hookrightarrow X$ , and the functor  $\Psi_{\pi \circ \tau, I}$  is an equivalence. Now define the closed subset  $|D| = |\mathcal{U}|^c \subset |X|$ , and take  $\iota : D \rightarrow X$  to denote the inclusion of a subscheme  $D$ , supported precisely on  $|D|$ , with coherent ideal sheaf  $\mathcal{J}$ . By Lemma 4.9, there is an  $n > 0$  such that the  $\mathcal{O}_3$ -ideal  $\widehat{\mathcal{J}}_3^n$  annihilates the kernel and cokernel of the map  $\mathcal{O}_3 \rightarrow \eta_* \mathcal{O}_{\widehat{\tau}^* \mathfrak{Z}}$ , where  $\eta : \widehat{\tau}^* \mathfrak{Z} \rightarrow \mathfrak{Z}$  is the adjunction map. Next, observe that the formal scheme  $\widehat{\tau}^* \mathfrak{Z}$  belongs to  $\mathbf{QF}_p(\widehat{X}/\widehat{S})$  and by hypothesis, there is an  $E$  in  $\mathbf{QF}_p(X'/S)$  such that  $\widehat{E} \cong \widehat{\tau}^* \mathfrak{Z}$ . Furthermore, Theorem 3.20, implies that we have isomorphisms  $\widehat{\tau}_! \widehat{\tau}^* \mathfrak{Z} \cong \widehat{\tau}_! \widehat{E} \cong (\widehat{\tau}_! E)$ . Setting  $Z' = \tau_! E$ , we obtain a finite and surjective map  $\eta : \widehat{Z}' \rightarrow \mathfrak{Z}$  such that the kernel and cokernel are annihilated by the  $\mathcal{O}_3$ -ideal  $\widehat{\mathcal{J}}_3^n$ . Since  $X$  is a noetherian scheme, we may now apply Lemma 4.7(2) to the map  $\eta : \widehat{Z}' \rightarrow \mathfrak{Z}$  and conclude that  $\mathfrak{Z}$  is effectivizable.  $\square$

With the dévissage technique for schemes fully developed, we now proceed to develop a dévissage technique for algebraic stacks.

**Lemma 4.11.** *Let  $X$  be a noetherian scheme,  $\pi : X' \rightarrow X$  a finite surjection, flat over an open subset  $\mathcal{U} \subset X$ . Let  $\pi^2 : X' \times_X X' \rightarrow X$  denote the induced morphism. Furthermore, let  $\mathcal{J}$  be a coherent  $\mathcal{O}_X$ -ideal with support  $|X| \setminus |\mathcal{U}|$ . Let  $\mathcal{J}'$  be another coherent  $\mathcal{O}_X$ -ideal, and let  $\widehat{X}$  (resp.  $\widehat{X}'$ ) denote the formal scheme obtained by completing  $X$  along the closed subset  $V(\mathcal{J})$  (resp.  $V(\mathcal{J}')$ ). Let  $(\mathfrak{Z} \rightarrow \widehat{X}) \in \mathbf{QF}(\widehat{X})$  and consider the diagram  $[\pi^2_!(\pi^2)^* \mathfrak{Z} \rightrightarrows \widehat{\pi}_! \widehat{\pi}^* \mathfrak{Z}]$ . Assume that this diagram has a uniform formal geometric colimit  $\widetilde{\mathfrak{Z}}$  (this is true in the case that  $\widehat{\pi}^* \mathfrak{Z}$  and  $(\pi^2)^* \mathfrak{Z}$  are effectivizable, by Corollary 2.19), so that there is an induced finite map  $\gamma_{\pi, \mathfrak{Z}} : \widetilde{\mathfrak{Z}} \rightarrow \mathfrak{Z}$ . Then, there is a  $k$  such that the kernel and cokernel of the map  $\gamma_{\pi, \mathfrak{Z}}^\sharp$  is annihilated by  $\mathcal{J}_3^k$ .*

*Proof.* Repeating verbatim the reductions in Lemma 4.9, it suffices to prove the result in the case that  $X = \operatorname{Spec} R$ , where  $R$  is a local and maximal-adically complete noetherian ring. Further, let  $R'$  be the finite  $R$ -algebra such that  $X' = \operatorname{Spec} R'$ . Set  $I = \Gamma(X, \mathcal{J})$  and  $J = \Gamma(X, \mathcal{J}')$ . We will prove the result by induction on  $\dim R = n$ . The case where  $n = 0$  is trivial (since  $R$  is local and artinian). Otherwise, assume that  $n > 0$ , and we have proven the result for all complete local rings  $S$  with  $\dim S < n$ . As in Lemma 4.9, this ensures

that the result has been proven for all noetherian schemes  $X$  such that all the local rings of  $X$  have  $\dim < n$ . Note that  $\mathfrak{Z} = \mathfrak{Z}^1 \amalg \mathfrak{Z}^2$ , where  $\mathfrak{Z}^1 \rightarrow \widehat{X}$  is finite, contains the special fiber, and  $\mathfrak{Z}^2$  misses the special fiber. In particular,  $\mathfrak{Z}^2 \rightarrow W := (\operatorname{Spec} R) - \{\mathfrak{m}\}$ , where  $\mathfrak{m}$  is the maximal ideal of  $R$ . Since  $R$  is local, all of the local rings of  $W$  have  $\dim < n$  and by the inductive hypothesis we can find a  $k_2$  such that  $\mathcal{J}_{\mathfrak{Z}^2}^{k_2}$  annihilates the kernel and cokernel of  $\gamma_{\pi, \mathfrak{Z}^2}^\#$ . Now, since the map  $\mathfrak{Z}^1 \rightarrow \widehat{X}$  is finite, then  $\mathfrak{Z}^1 = \operatorname{Spf}_{\operatorname{IT}} T$ , where  $T$  is a finite  $R$ -algebra. By definition, if we set  $\widetilde{T} = \ker(T \otimes_R R' \xrightarrow{h} T \otimes_R R' \otimes_R R')$ , where  $h(t \otimes r) = t \otimes 1 \otimes r - t \otimes r \otimes 1$ , then  $\widetilde{\mathfrak{Z}}^1 = \operatorname{Spf}_{\operatorname{IT}} \widetilde{T}$ . Let  $f \in J$ , then  $R_f \rightarrow R'_f$  is flat. By flat descent for modules, we conclude that  $T_f \rightarrow \widetilde{T}_f$  is an isomorphism. Thus, the kernel and cokernel of  $\gamma_{\pi, \mathfrak{Z}^1}^\# : T \rightarrow \widetilde{T}$  is annihilated by some fixed power of  $f$ . Since  $J$  is coherent, then we conclude that there is a  $k_1$  such that  $J^{k_1}$  annihilates the kernel and cokernel of  $\gamma_{\pi, \mathfrak{Z}^1}^\#$ . It is now clear that taking  $k = \max\{k_1, k_2\}$  has the property that  $\mathcal{J}_{\mathfrak{Z}}^k$  annihilates the kernel and cokernel of  $\gamma_{\pi, \mathfrak{Z}}^\#$ .  $\square$

**Proposition 4.12** (Generically finite flat dévissage). *Let  $\pi : X \rightarrow S$  be a finite type morphism of noetherian algebraic stacks with schematic diagonal. For any  $S$ -map  $p : Y \rightarrow X$ , define  $p^2 : Y \times_X Y \rightarrow X$ . Suppose that for any closed substack  $\iota_V : V \hookrightarrow X$  there is a finite and generically flat (on  $V$ ) morphism  $\tau_V : V' \rightarrow V$  where  $\Psi_{\pi \circ \tau_V, I}$  and  $\Psi_{\pi \circ \tau_V^2, I}$  are equivalences. Then  $\Psi_{\pi, I}$  is an equivalence.*

*Proof.* The principle of noetherian induction implies that we may assume that  $\Psi_{\pi \circ \iota_V, I}$  is an equivalence for any closed substack  $\iota_V : V \hookrightarrow X$  with  $|V| \subsetneq |X|$ . By Lemma 4.2, it suffices to show that  $\Psi_{\pi, I}$  is essentially surjective.

By assumption, there is a finite map  $\tau : X^1 \rightarrow X$  such that over a dense open  $U \subset X$  the induced map  $\tau : \tau^{-1}U \rightarrow U$  is flat with the property that  $\Psi_{\pi \circ \tau, I}$  and  $\Psi_{\pi \circ \tau^2, I}$  are equivalences. It will be convenient to set  $\tau^1 := \tau$ ,  $X^2 := X^1 \times_X X^1$  and we then have  $\tau^2 : X^2 \rightarrow X$ . Let  $(Z_n)_{n \geq 0} \in \varprojlim_n \mathbf{QF}_p(X_n/S_n)$  and for each  $n \geq 0$  and  $i = 1, 2$  we set  $Z_n^i = Z_n \times_{X_n} X_n^i$ , then for  $i = 1, 2$  we have  $(Z_n^i)_{n \geq 0} \in \varprojlim_n \mathbf{QF}_p(X_n^i/S_n)$ . By assumption, for  $i = 1, 2$  there is a unique  $(Z^i \rightarrow X^i) \in \mathbf{QF}_p(X^i/S)$  such that  $\Psi_{\pi \circ \tau^i, I}(Z^i \rightarrow X^i) = (Z_n^i)_{n \geq 0}$ . Note that from the two finite projection maps  $s, t : X^2 \rightarrow X^1$  we obtain a coequalizer diagram  $[Z^2 \rightrightarrows Z^1]$  in  $\mathbf{QF}(X)$ . By Theorem 2.10, this diagram has a uniform categorical and uniform geometric coequalizer  $(C \rightarrow X)$  in  $\mathbf{QF}(X)$ . Note that since  $Z^1 \rightarrow C$  is finite and surjective and  $Z^1$  is  $S$ -proper, then  $C$  is  $S$ -proper, thus  $(C \rightarrow X) \in \mathbf{QF}_p(X/S)$ .

Let  $p^1 : V^1 \rightarrow X$  be a smooth surjection from a scheme  $V^1$ . Consider an integer  $j \geq 1$  and let  $V^j$  be the  $j$ -fold fiber product of  $V^1$  over  $X$  and since  $X$  has schematic diagonal,  $V^j$  is an  $S$ -scheme. The  $V^j$ -scheme  $C_{V^j}$  is the categorical and geometric coequalizer of the diagram  $[Z_{V^j}^2 \rightrightarrows Z_{V^j}^1]$  in  $\mathbf{QF}(V^j)$ . For each  $n \geq 0$ , the morphism  $(C_{V^j})_n \rightarrow V_n^1$  comes with descent data for the smooth covering  $V_n^1 \rightarrow X_n$ . We also note here that for each  $n$ , the morphism  $(Z_n)_{V^1} \rightarrow V_n^1$  comes equipped with descent data for the smooth covering  $p_n^1 : V_n^1 \rightarrow X_n$ .

Define the formal  $\widehat{V^j}$ -scheme  $\mathfrak{Z}_{V^j} = \varprojlim_n (Z_n \times_X V^j) \in \mathbf{QF}(\widehat{V^j})$  and let  $\sigma_j : \mathfrak{Z}_{V^j} \rightarrow \widehat{V^j}$  denote the structure map. By Theorem 2.18,  $\widehat{C_{V^j}}$  is the uniform categorical and uniform formal geometric coequalizer of the diagram  $[\widehat{Z_{V^j}^2} \rightrightarrows \widehat{Z_{V^j}^1}]$  in  $\mathbf{QF}(\widehat{V^j})$ , and whence there is a unique finite morphism of formal  $\widehat{V^j}$ -schemes  $\phi^j : \widehat{C_{V^j}} \rightarrow \mathfrak{Z}_{V^j}$ .



Consider the closed subset  $|D| = |U|^c \subset |X|$ , and fix a coherent sheaf of ideals  $\mathcal{I} \ll \mathcal{O}_X$  with support  $|D|$ . By Lemma 4.11 there is a  $k > 0$  for  $j = 1, 2$ , such that  $(\mathcal{I}_{V^j})_{\mathfrak{Z}_{V^j}}^k$  annihilates the kernel and cokernel of  $(\phi^j)^\#$ . By Lemma 4.7(2), we thus conclude that  $\mathfrak{Z}_{V^j}$  is effectivizable for  $j = 1, 2$ . Thus, we may find  $(W^j \rightarrow V^j) \in \mathbf{QF}(V^j)$  such that  $\widehat{W^j} \cong \mathfrak{Z}_{V^j}$ . By smooth descent of quasi-finite and separated morphisms, we may conclude that there is a quasi-finite and separated scheme  $(Z \rightarrow X) \in \mathbf{QF}(X)$  such that  $Z_{V^j} \cong W^j$  for  $j = 1, 2$ . Further, since smooth descent commutes with arbitrary fiber product, we conclude that  $Z \times_X X_n \cong Z_n$ . Finally, to see that  $(Z \rightarrow X) \in \mathbf{QF}_p(X)$ , we simply note that the  $X$ -map  $\phi : C \rightarrow Z$  is finite and surjective (this is verified smooth locally on  $X$ ). Thus, we have proved that  $(Z_n)_{n \geq 0} \in \varinjlim_n \mathbf{QF}_p(X_n/S_n)$  lies in the essential image of  $\Psi_{\pi, I}$ .  $\square$

We may finally prove Theorem 4.5.

*Proof of Theorem 4.5.* By Lemma 4.2, it remains to show that the functor  $\widetilde{\Psi}_{\pi, I}$  is essentially surjective, which we divide into a number of cases.

**Basic Case.** Here, we assume that we have a finite type morphism of schemes  $\pi : X \rightarrow S$  which factors as  $X \xrightarrow{p} Y \xrightarrow{q} S$  where  $p$  is étale and  $q$  is projective. Now, suppose that  $(\mathfrak{Z} \xrightarrow{\sigma} \widehat{X}) \in \mathbf{QF}_p(\widehat{X}/\widehat{S})$ , then we observe that the composition of morphisms  $\mathfrak{Z} \rightarrow \widehat{X} \rightarrow \widehat{Y}$  is quasi-finite and proper, since  $Y$  is separated and  $\mathfrak{Z}$  is  $\widehat{S}$ -proper. Hence, the morphism  $\mathfrak{Z} \rightarrow \widehat{Y}$  is finite, and as  $q : Y \rightarrow S$  is projective, by [EGA, III, 5.4.4], there is a finite  $Y$ -scheme  $Z$  such that in  $\mathbf{QF}_p(\widehat{Y}/\widehat{S})$ ,  $(\widehat{Z} \rightarrow \widehat{Y}) \cong (\mathfrak{Z} \rightarrow \widehat{Y})$ . It now remains to construct an  $S$ -morphism  $s : Z \rightarrow X$  such that  $\widehat{s} = \sigma$ . To do this, it suffices to construct a section to the morphism  $X \times_Y Z \rightarrow Z$  which restricts to the section given by the morphism  $\sigma_n : Z_n \rightarrow X_n$  for all  $n$ . The hypotheses on  $X \rightarrow Y$  are all preserved by base change, so we conclude that  $X \times_Y Z \rightarrow Z$  is quasi-compact and étale. Hence, the étale sheaf on  $Z$  given by  $U \mapsto \mathrm{Hom}_Z(U, X \times_Y Z)$  is constructible. Since we have a section  $Z_0 \rightarrow X_0 \times_{Y_0} Z_0$ , by [Mil80, Thm. VI.2.1], we obtain a unique section  $Z \rightarrow X \times_Y Z$  which restricts to the one we started with (hence to all other  $\sigma_n$ 's).

**Quasi-compact and schematic case.** Here, we assume that we have an arbitrary finite type morphism of schemes  $\pi : X \rightarrow S$ . Note that any closed subscheme  $V$  of  $X$  is of finite type. In particular, by [RG71, Cor. 5.7.13], there is always a diagram:

$$\begin{array}{ccc} & V' & \\ \swarrow & & \searrow \\ V & & P \\ \searrow & & \swarrow \\ & S & \end{array},$$

where  $V' \rightarrow V$  is proper and birational on  $V$ ,  $P \rightarrow S$  is projective, and  $V' \rightarrow P$  is étale. By the basic case considered above, we know that  $\Psi_{V' \rightarrow S, I}$  is an equivalence. Thus, we have met the criteria to apply Proposition 4.10 and we conclude that  $\Psi_{\pi, I}$  is an equivalence for any finite type morphism of schemes  $\pi : X \rightarrow S$ .

**Quasi-compact with quasi-finite and separated diagonal case.** Now we assume that we have a finite type morphism of algebraic stacks  $\pi : X \rightarrow S$  with quasi-finite and separated diagonal. By Proposition 4.12 and the case for schemes already proved, it suffices to show that for any closed substack  $V \hookrightarrow X$ , there is a finite surjection  $V' \rightarrow V$  from a scheme  $V'$ , which is flat over a dense open of  $V$ . This is [Ryd09, Thm. B].

**General case.** Suppose that we have a locally of finite type morphism of locally noetherian algebraic stacks  $\pi : X \rightarrow S$  which has quasi-compact and separated diagonal, then it

remains to show that the functor  $\tilde{\Psi}_{\pi, I}$  is essentially surjective. Let  $Y \hookrightarrow X$  be the open substack of  $X$  with quasi-finite and separated diagonal. If  $(Z_n)_{n \geq 0} \in \varprojlim_n \mathbf{QF}_{p, \text{fin}} \Delta(X_n/S_n)$ , then for each  $n \geq 0$ , the structure map  $Z_n \rightarrow X$  factors uniquely through  $Y$ . Hence, it suffices to prove that the functor  $\tilde{\Psi}_{\pi, I}$  is essentially surjective in the case that the morphism of locally noetherian algebraic stacks  $\pi : X \rightarrow S$  has quasi-finite and separated diagonal. Let  $O_X$  denote the set of quasi-compact open substacks  $X$ , which is ordered by inclusion, and we note that  $\{W \times_X Z_0\}_{W \in O_X}$  is an open cover of the quasi-compact topological space  $|Z_0|$ . Thus, there is a  $W \in O_X$  such that the canonical  $X$ -morphism  $W \times_X Z_0 \rightarrow Z_0$  is an isomorphism. In particular, we conclude that the map  $Z_0 \rightarrow X$  factors uniquely as  $Z_0 \rightarrow W \hookrightarrow X$ . Since the open immersion  $W \hookrightarrow X$  is étale, it follows that the maps  $Z_n \rightarrow X$  also factor compatibly through  $W$  and thus  $(Z_n)_{n \geq 0} \in \varprojlim_n \mathbf{QF}_p(W_n/S_n)$ , which reduces us to proving that the functor  $\Psi_{\pi, I}$  is essentially surjective in the case of a finite type morphism of noetherian algebraic stacks  $\pi : X \rightarrow S$  with quasi-finite and separated diagonal, which we've already shown.  $\square$

## 5. THE HILBERT STACK

In this section, we will use Theorem 4.5 and Lurie's Representability Theorem [Lur04, Thm. 7.1.6] to prove Theorem 2. For all definitions and conventions for derived algebraic geometry, we follow [Lur04]. For a simplicial commutative ring  $R$ , define the  $\infty$ -category  $\mathcal{SCR}_R$  to be the  $\infty$ -category of simplicial commutative  $R$ -algebras with the model structure inherited from the underlying simplicial sets. Let  $\mathcal{S}$  be the  $\infty$ -category of spaces. The main content of Theorem 2 will be implied by the next result, which will take most of this section, and Appendix A, to prove.

**Theorem 5.1.** *Let  $R$  be a discrete ring, set  $S = \text{Spec } R$ , and consider a locally finitely presented morphism of algebraic stacks  $\pi : X \rightarrow S$  with quasi-compact and separated diagonal, then the functor  $\mathcal{HS}_{X/S}^{\text{der}} : \mathcal{SCR}_R \rightarrow \mathcal{S}$  is a derived 1-stack which is almost finitely presented over  $R$ .*

The functor  $\mathcal{HS}_{X/S}^{\text{der}}$  is the natural extension of the fibered category  $\underline{\text{HS}}_{X/S}$ , which is the object of interest in Theorem 2, to the derived setting. Since it will be necessary, however, to vary the definitions in [Lur04] to prove Theorem 2 using the language of derived algebraic geometry, we will postpone the precise definition of  $\mathcal{HS}_{X/S}^{\text{der}}$ , and the proof of Theorem 5.1 until later in this section. For the moment, we will busy ourselves by proving that the moduli functor  $\mathcal{HS}_{X/S}^{\text{der}}$  possesses a cotangent complex, which will be an efficient way of packaging the deformation theory of the objects of the (derived) Hilbert stack. We begin by defining some notions that are related to the deformation theory of derived stacks, as well as some very general moduli functors.

For a simplicial ring  $A$  and integer  $n \geq 0$ , let the simplicial ring  $\tau_{\leq n} A$  denote the  $n$ -truncation of  $A$ , which is obtained by filling in all cells in degree  $> n$ . By [Lur04, Prop. 5.4.4], the notion of truncation can be extended to derived stacks, and we write the  $n$ -truncation of a derived stack  $X$  as  $\tau_{\leq n} X$ . Denote the 2-category of algebraic stacks by  $\mathbf{AlgStk}$ , then there is a fully faithful embedding of  $\mathbf{AlgStk}$  into the  $\infty$ -category of  $\mathcal{S}$ -valued functors on  $\mathcal{SCR}$ . The essential image of  $\mathbf{AlgStk}$  is precisely those derived 1-stacks which are 0-truncated. We observe that the full subcategory of algebraic spaces in  $\mathbf{AlgStk}$  has essential image, under the aforementioned embedding, the derived 0-stacks which are 0-truncated.

**Definition 5.2.** This is lifted from [Lur04, Defn. 3.4.1]. Fix a simplicial ring  $T$ . A functor  $F : \mathcal{SCR}_T \rightarrow \mathcal{S}$  is **cohesive** if for any  $R$ -algebras  $A, B, C$  together with closed immersions

$\mathrm{Spec} C \rightarrow \mathrm{Spec} A$ ,  $\mathrm{Spec} C \rightarrow \mathrm{Spec} B$ , the natural map:

$$F(A \times_C B) \rightarrow F(A) \times_{F(C)} F(B)$$

is an equivalence. The functor  $F$  is **nilcomplete** if for any  $R$ -algebra  $A$ , the canonical map  $F(A) \rightarrow \varprojlim_n F(\tau_{\leq n} A)$  is an equivalence. We say that a morphism of functors  $F \rightarrow G$  from  $\mathcal{SCR}_{R/} \rightarrow \mathcal{S}$  is **cohesive** (resp. **nilcomplete**) if for any  $R$ -algebra  $S$ , and any  $\eta \in G(S)$ , then the fiber functor  $F_\eta : \mathcal{SCR}_{S/} \rightarrow \mathcal{S}$  is cohesive (resp. nilcomplete).

In [Lur04, Thm. 5.6.4], it is shown that any derived stack is cohesive. This simply means that you can glue objects in your stack along closed immersions of affine derived schemes. A weaker notion than cohesiveness of a functor is **infinitesimal cohesiveness**, which only requires the ability to glue along nilimmersions of affine derived schemes. Infinitesimal cohesiveness of a functor is an analog of the Schlessinger-Rim criteria in the derived setting (cf. [Lur04, §6.2]), and will feature in proving Theorem 5.1. Since cohesiveness implies infinitesimal cohesiveness, and the functor  $\mathcal{HS}_{X/S}^{\mathrm{der}}$  will be proven to be cohesive, it will be unnecessary for us to mention infinitesimal cohesiveness again.

In [Lur04, Prop. 5.3.7], it is shown that any derived stack is nilcomplete. In concrete terms, nilcompleteness of a functor means that the values of the functor on a simplicial ring with cells in arbitrarily high degree are always lifted from compatible values of the functor on truncations. This is similar to the ability to effectivize formal deformations, it is just in the derived direction.

**Definition 5.3.** Fix a simplicial commutative ring  $R$ . For a morphism of derived  $R$ -stacks  $U \rightarrow \mathrm{Spec} R$ , define the functor  $\mathbf{DerStk}_U : \mathcal{SCR}_{R/} \rightarrow \mathcal{S}$  as follows: for any simplicial commutative  $R$ -algebra  $A$ ,  $\mathbf{DerStk}_U(A)$  is the  $\infty$ -groupoid of relative derived stacks  $Z \rightarrow U \times_{\mathrm{Spec} R} \mathrm{Spec} A$  such that the composition  $Z \rightarrow \mathrm{Spec} A$  is an almost finitely presented and flat morphism of derived stacks.

The moduli functor just defined, subsumes the Hilbert stack. Indeed, if  $R$  is a discrete ring and  $X \rightarrow \mathrm{Spec} R$  is a morphism of algebraic stacks, then for any discrete  $R$ -algebra  $A$ , the  $\infty$ -groupoid  $\mathbf{DerStk}_X(A)$  consists of those morphisms of derived stacks  $Z \rightarrow X \otimes_R A$  such that the composition  $f : Z \rightarrow \mathrm{Spec} A$  is almost finitely presented and flat. Since  $A$  is discrete, and the morphism  $f$  is flat, then  $Z$  is discrete. Thus, we conclude that if  $Z$  is a 1-stack, then  $Z$  is a locally finitely presented algebraic stack. Hence, we have a fully faithful embedding of groupoids  $\mathbf{HS}_{X/S}(\mathrm{Spec} A) \rightarrow \mathbf{DerStk}_X(A)$ . This added flexibility will be useful for proving Theorem 5.1.

**Lemma 5.4.** Fix a simplicial commutative ring  $R$ , and let  $U$  be a derived  $R$ -stack. Then, the functor  $\mathbf{DerStk}_U$  is cohesive and nilcomplete.

*Proof.* The cohesiveness is immediate from Theorem A.1, the nilcompleteness follows from the results of [Lur04, §5.4].  $\square$

Given a morphism of derived  $R$ -stacks  $U \rightarrow V$ , there is a morphism of moduli functors  $F_{U/V} : \mathbf{DerStk}_U \rightarrow \mathbf{DerStk}_V$ . In particular, there is the map  $F_{U/\mathrm{Spec} R} : \mathbf{DerStk}_U \rightarrow \mathbf{DerStk}_{\mathrm{Spec} R}$ . The following computation is identical to [Lur04, Prop. 8.1.2], so we omit it.

**Lemma 5.5.** Fix a simplicial commutative ring  $R$ , write  $S = \mathrm{Spec} R$  and let  $U$  be a derived  $R$ -stack. For a simplicial commutative  $R$ -algebra  $A$ , and a connective  $A$ -module  $M$ , write  $T = \mathrm{Spec} A$  and  $T[M] = \mathrm{Spec}(A \oplus M)$ , then the fiber of the natural transformation:

- (1)  $\mathbf{DerStk}_S(A \oplus M) \rightarrow \mathbf{DerStk}_S(A)$  over  $\eta = (X \xrightarrow{f} \mathrm{Spec} A) \in \mathbf{DerStk}_S(A)$  is naturally equivalent to  $\mathrm{Hom}_{\mathrm{QC}_X}(L_{X/\mathrm{Spec} A}, f^* M[1])$ .

- (2)  $\mathbf{DerStk}_{\mathbf{U}}(A \oplus M) \rightarrow \mathbf{DerStk}_{\mathbf{U}}(A) \times_{\mathbf{DerStk}_S(A)} \mathbf{DerStk}_S(A \oplus M)$  over  $(Z \xrightarrow{s} \mathbf{U}_T, Z' \rightarrow T[M], \gamma : Z \simeq Z' \times_{T[M]} T) \in \mathbf{DerStk}_{\mathbf{U}}(A) \times_{\mathbf{DerStk}_S(A)} \mathbf{DerStk}_S(A \oplus M)$  is naturally equivalent to

$$\mathrm{Hom}_{\mathrm{QC}_Z}(L_{Z/\mathbf{U}_T}, g^*M),$$

where the map  $g$  is the composition  $Z \rightarrow \mathbf{U}_T \rightarrow T$ .

The notion of properness given in [Lur04, Defn. 5.5.1], is too restrictive for our purposes (it applies to derived Deligne-Mumford stacks), so we will introduce a slight variant of this notion that is sufficient for the purposes of this paper.

**Definition 5.6.** Let  $P$  be a property of morphisms of algebraic stacks, which is local on the target for the smooth topology, and is preserved by arbitrary base change. We say that a relative 1-stack  $F' \rightarrow F$  is  $P$  if for any discrete ring  $R$ , and any morphism  $\mathrm{Spec} R \rightarrow F$ , the induced morphism of algebraic stacks  $\tau_{\leq 0}(\mathrm{Spec} R \times_F F') \rightarrow \mathrm{Spec} R$  has the property  $P$ .

We will only use this definition for “quasi-finite”, “proper”, and “finite” morphisms. Using the results of [Ols05], we observe that all of the coherence results from [Lur04, §5.5] for this variant of a proper morphism (with the addition of the property almost finitely presented) go through without change. It is important to note that the induced notion of flatness is *incorrect* in the derived setting. For this paper, we preserve the notion of flatness used in [Lur04].

**Definition 5.7.** Fix a discrete ring  $R$  and set  $S = \mathrm{Spec} R$ . For a morphism of algebraic stacks  $\pi : X \rightarrow S$ , define the functor  $\mathcal{H}_{X/S}^{\mathrm{der}} : \mathcal{S}\mathcal{C}\mathcal{R}_R \rightarrow \mathcal{S}$  as follows: for a simplicial  $R$ -algebra  $A$ ,  $\mathcal{H}_{X/S}^{\mathrm{der}}(A)$  is the  $\infty$ -groupoid of almost finitely presented, quasi-finite, relative algebraic spaces  $s : Z \rightarrow X \times_S \mathrm{Spec} A$  such that the composition  $\pi_{\mathrm{Spec} A} \circ s : Z \rightarrow \mathrm{Spec} A$  is almost finitely presented, flat, proper, and with finite diagonal.

This next result is fundamental.

**Lemma 5.8.** Fix a discrete ring  $R$  and set  $S = \mathrm{Spec} R$ . For a morphism of algebraic stacks  $\pi : X \rightarrow S$ , the functor  $\mathcal{H}_{X/S}^{\mathrm{der}}$  is cohesive and nilcomplete.

*Proof.* From Lemma 5.4, the nilcompleteness is clear. For the cohesiveness, we note that given a diagram of affine derived schemes  $[T_1 \xleftarrow{t_1} T_3 \xrightarrow{t_2} T_2]$ , where the  $t_i$  are closed immersions, and a diagram of flat, proper and almost finitely presented derived 1-stacks  $[Z_1 \xleftarrow{u_1} Z_3 \xrightarrow{u_2} Z_2]$  lying over the previous diagram, then by Theorem A.1, we have a flat and almost finitely presented morphism of the categorical pushouts  $v : Z_1 \amalg_{Z_3} Z_2 \rightarrow T_1 \amalg_{T_3} T_2$ . It remains to show that  $v$  is a proper morphism of 1-stacks. Thus, it suffices to show that  $\tau_{\leq 0} v : \tau_{\leq 0}(Z_1 \amalg_{Z_3} Z_2) \rightarrow \tau_{\leq 0}(T_1 \amalg_{T_3} T_2)$  is a proper morphism of algebraic stacks. There is a commutative diagram of algebraic stacks:

$$\begin{array}{ccc} \tau_{\leq 0} Z_1 \amalg \tau_{\leq 0} Z_2 & \xrightarrow{d'} & \tau_{\leq 0}(Z_1 \amalg_{Z_3} Z_2) \\ \downarrow & & \downarrow \\ \tau_{\leq 0} T_1 \amalg \tau_{\leq 0} T_2 & \xrightarrow{d} & \tau_{\leq 0}(T_1 \amalg_{T_3} T_2). \end{array}$$

By Lemma A.5 the maps  $d$  and  $d'$  are finite surjections. Hence, the algebraic stack  $\tau_{\leq 0}(Z_1 \amalg_{Z_3} Z_2)$  is finitely dominated by a proper  $\tau_{\leq 0}(T_1 \amalg_{T_3} T_2)$ -stack, and is thus proper over  $\tau_{\leq 0}(T_1 \amalg_{T_3} T_2)$ .  $\square$

Fix a discrete ring  $R$  and set  $S = \operatorname{Spec} R$ . If  $\pi : X \rightarrow S$  is a morphism of algebraic stacks and  $A$  is a discrete  $R$ -algebra, then  $\mathcal{H}\mathcal{S}_{X/S}^{\operatorname{der}}(A)$  is the  $\infty$ -groupoid with objects the almost finitely presented, quasi-finite, relative algebraic spaces  $s : Z \rightarrow X \times_S \operatorname{Spec} A$  such that the composition  $\pi_{\operatorname{Spec} A} \circ s : Z \rightarrow \operatorname{Spec} A$  is almost finitely presented, flat, and proper. Since  $\pi_{\operatorname{Spec} A} \circ s$  is flat, and  $\operatorname{Spec} A$  is 0-truncated, it follows that  $Z$  is 0-truncated. Hence, we see that  $\pi_{\operatorname{Spec} A} \circ s : Z \rightarrow \operatorname{Spec} A$  is a morphism of algebraic stacks which is finitely presented, flat, proper and with finite diagonal. In particular, the morphism  $s : Z \rightarrow X \times_S \operatorname{Spec} A$  is a representable morphism of algebraic stacks which is quasi-finite. If  $\pi : X \rightarrow S$  has separated diagonal, the morphism  $s$  is automatically separated. In particular, there is an equivalence  $\underline{\operatorname{HS}}_{X/S}(\operatorname{Spec} A) \rightarrow \mathcal{H}\mathcal{S}_{X/S}^{\operatorname{der}}(A)$  for any discrete  $R$ -algebra  $A$ . It will still be convenient to work in greater generality than this for the moment.

**Definition 5.9.** Fix a simplicial commutative ring  $R$ , and a derived  $R$ -stack  $U$ . Define the functor  $\mathbf{DerStk}_U^{1,\operatorname{Prop}} : \mathcal{SCR}_R \rightarrow \mathcal{S}$  to be that which assigns to each simplicial  $R$ -algebra  $A$ , the  $\infty$ -groupoid of morphisms of derived stacks  $Z \rightarrow U \times_{\operatorname{Spec} R} \operatorname{Spec} A$ , such that the composition  $Z \rightarrow U \times_{\operatorname{Spec} R} \operatorname{Spec} A \rightarrow \operatorname{Spec} A$  is an almost finitely presented, flat and proper relative 1-stack.

The next result is clear.

**Lemma 5.10.** Fix a simplicial commutative ring  $R$ , write  $S = \operatorname{Spec} R$ , and consider a morphism of algebraic stacks  $X \rightarrow S$ . Then the natural transformation of functors  $\mathcal{H}\mathcal{S}_{X/S}^{\operatorname{der}} \rightarrow \mathbf{DerStk}_X^{1,\operatorname{Prop}}$  has trivial cotangent complex.

For a simplicial commutative ring  $A$ , we let  $\mathcal{M}_A$  denote the stable  $\infty$ -category of  $A$ -modules. For details, we refer the reader to [Lur04, §2]. If  $A$  is a discrete ring, the  $\infty$ -category  $\mathcal{M}_A$  has objects the complexes of  $A$ -modules and the morphisms are topological spaces whose homotopy groups correspond to homotopies between morphisms of chain complexes. Passing to the homotopy category of  $\mathcal{M}_A$  produces the derived category of  $A$ -modules. We require this terminology for the following

**Proposition 5.11.** Fix a simplicial commutative ring  $R$ , and a derived  $R$ -stack  $U$ , then the functor  $\mathbf{DerStk}_U^{1,\operatorname{Prop}}$  has a cotangent complex. Thus, if  $U \rightarrow \operatorname{Spec} R$  is a morphism of algebraic stacks, then the functor  $\mathcal{H}\mathcal{S}_{U/\operatorname{Spec} R}^{\operatorname{der}}$  has a cotangent complex.

*Proof.* The assertion for  $\mathcal{H}\mathcal{S}_{U/\operatorname{Spec} R}^{\operatorname{der}}$  follows from the first claim, combined with Lemma 5.10 and [Lur04, Prop. 3.2.12]. Now, write  $S = \operatorname{Spec} R$ , and we have a sequence of natural transformations:

$$\mathbf{DerStk}_U^{1,\operatorname{Prop}} \xrightarrow{F_{U/S}} \mathbf{DerStk}_S^{1,\operatorname{Prop}} \longrightarrow S.$$

Thus, by [Lur04, Prop. 3.2.12] it suffices to show that the functor  $\mathbf{DerStk}_S^{1,\operatorname{Prop}}$  and the natural transformation  $F_{U/S}$  have cotangent complexes. For the first claim, let  $C \in \mathcal{SCR}_R$  and fix a connective  $C$ -module  $M$ . Further, fix  $\eta \in \mathbf{DerStk}_S^{1,\operatorname{Prop}}(C)$  and observe that  $\eta$  corresponds to an almost finitely presented, flat, and proper derived 1-stack  $f : Z \rightarrow \operatorname{Spec} C$ . By Lemma 5.5(1), the fiber of  $\mathbf{DerStk}_S^{1,\operatorname{Prop}}(C \oplus M) \rightarrow \mathbf{DerStk}_S^{1,\operatorname{Prop}}(C)$  over  $\eta$  is given by  $\operatorname{Hom}_{\operatorname{QC}_Z}(L_{Z/\operatorname{Spec} C}, f^*M[1])$ . Since  $f$  is proper and almost finitely presented, by [Lur04, Cor. 5.5.8], it follows that the functor on connective  $C$ -modules

$$M \mapsto \operatorname{Hom}_{\operatorname{QC}_Z}(L_{Z/\operatorname{Spec} C}, f^*M[1])$$

is corepresentable by an almost connective  $C$ -module  $L_{\mathbf{DerStk}_S^{1,\operatorname{Prop}}}(\eta) \in \mathcal{M}_C$ . If  $\psi : C \rightarrow C'$  is a morphism in  $\mathcal{SCR}_R$ , then to show that  $\mathbf{DerStk}_S^{1,\operatorname{Prop}}$  has a cotangent complex, it

suffices to show that the canonical map

$$\phi_\psi : L_{\mathbf{DerStk}_S^{1, \text{Prop}}}(\eta) \otimes_C C' \rightarrow L_{\mathbf{DerStk}_S^{1, \text{Prop}}}(\psi_* \eta)$$

is an equivalence. Write  $\psi_* \eta$  as  $f' : Z' \rightarrow \text{Spec } C'$  and let  $g : Z' \rightarrow Z$  be the natural map, then we have a functorial equivalence  $g^* L_{Z/\text{Spec } C} \rightarrow L_{Z'/\text{Spec } C'}$ . Thus, for a connective  $C'$ -module  $M'$  we have functorial equivalences

$$\begin{aligned} \text{Hom}_{\mathcal{M}_C}(L_{\mathbf{DerStk}_S^{1, \text{Prop}}}(\psi_* \eta), M') &\simeq \text{Hom}_{\text{QC}_Z}(L_{Z'/\text{Spec } C'}, f'^* M'[1]) \\ &\simeq \text{Hom}_{\text{QC}_Z}(g^* L_{Z/\text{Spec } C}, f'^* M'[1]) \simeq \text{Hom}_{\text{QC}_Z}(L_{Z/\text{Spec } C}, g_* f'^* M'[1]) \\ &\simeq \text{Hom}_{\text{QC}_Z}(L_{Z/\text{Spec } C}, f^* \psi_* M'[1]) \simeq \text{Hom}_{\mathcal{M}_C}(L_{\mathbf{DerStk}_S^{1, \text{Prop}}}(\eta), \psi_* M'[1]) \\ &\simeq \text{Hom}_{\mathcal{M}_{C'}}(L_{\mathbf{DerStk}_S^{1, \text{Prop}}}(\eta) \otimes_C C', M'). \end{aligned}$$

Hence, we conclude that the canonical map  $\phi_\psi$  is an equivalence, and that the functor  $\mathbf{DerStk}_S^{1, \text{Prop}}$  has a cotangent complex. Combining the technique of proof above, with Lemma 5.5(2), we also obtain that the natural transformation  $F_{U/S}$  possesses a cotangent complex, thus we're done.  $\square$

The following result is related to what appears in [Lur04, §5.4], and will be useful shortly.

**Lemma 5.12.** *Let  $\{S_i\}_{i \in I}$  be an inverse system of bounded derived stacks such that the transition maps  $S_i \rightarrow S_{i'}$  are affine for all  $i \geq i' \geq i_0 \in I$ . Fix a bounded and locally finitely presented to order  $m$  morphism  $f_{i_0} : X_{i_0} \rightarrow S_{i_0}$  of derived relative  $n$ -stacks. For  $i \geq i_0$  set  $f_i := f_{i_0} \times_{S_{i_0}} S_i$ . Let  $S = \varprojlim_i S_i$  and  $f := f_{i_0} \times_{S_{i_0}} S$ . If  $\tau_{\leq m} f : \tau_{\leq m} X \rightarrow \tau_{\leq m} S$  has one of the following properties:*

- (1) *flat; or*
- (2) *relative  $\mathcal{L}$ -stack; or*
- (3) *quasi-finite; or*
- (4) *proper; or*
- (5) *finite,*

*then there is an  $\alpha \geq i_0$  such that  $\tau_{\leq m} f_\beta : \tau_{\leq m} X_\beta \rightarrow \tau_{\leq m} S_\beta$  has the same property that  $\tau_{\leq m} f$  has for all  $\beta \geq \alpha$ .*

*Proof.* We note that (1) is smooth local on the source and target of  $f_{i_0}$ , so we immediately reduce to the case where  $S_{i_0} = \text{Spec } R_{i_0}$ ,  $X_{i_0} = \text{Spec } T_{i_0}$  and  $f_{i_0} : X_{i_0} \rightarrow S_{i_0}$  is induced, up to order  $m$ , from a finitely presented homomorphism of simplicial commutative rings  $R_{i_0} \rightarrow T_{i_0}$ . Since  $S_i \rightarrow S_{i'}$  is affine for  $i \geq i' \geq i_0$ , we are completely reduced to the affine case. For  $i \geq i_0$ , let the simplicial commutative ring  $R_i$  (resp.  $T_i$ ) be such that  $\text{Spec } R_i = S_i$  (resp.  $\text{Spec } T_i = X_i$ ). Let  $R = \varinjlim_{i \geq i_0} R_i$  and  $T = \varinjlim_{i \geq i_0} T_i$ .

Since  $\pi_0(R) \rightarrow \pi_0(T)$  is flat, by [EGA, IV, 11.2.6], there is an  $i_1 \geq i_0$  and a discrete, flat, finitely presented  $\pi_0(R_{i_1})$ -algebra  $B_{i_1}$  such that there is an equivalence  $B_{i_1} \otimes_{\pi_0(R_{i_1})} \pi_0(R) \simeq \pi_0(T)$ . For  $i \geq i_1$ , let  $B_i = \pi_0(R_i) \otimes_{\pi_0(R_{i_1})} B_{i_1}$ , then  $B_i$  is a discrete, flat and finitely presented  $\pi_0(R_i)$ -algebra, and  $\varinjlim_{i \geq i_1} B_i \simeq \pi_0(T)$ . Also, for  $i \geq i_1$ , let  $C_i$  be the finitely presented  $\pi_0(R_i)$ -algebra  $T_i \otimes_{R_i} \pi_0(R_i)$ . Note that by the flatness of  $\tau_{\leq m} R \rightarrow \tau_{\leq m} T$ , we have equivalences

$$\tau_{\leq m}(\varinjlim_i C_i) \simeq \tau_{\leq m}(T \otimes_R \pi_0(R)) \simeq (\tau_{\leq m} T) \otimes_{\tau_{\leq m} R} \pi_0(R) \simeq \pi_0(T) \simeq \varinjlim_i B_i.$$

Thus by [Lur04, Prop. 5.4.9], there is an  $i_2 \geq i_1$  and a homomorphism  $B_{i_2} \rightarrow C_{i_2}$  inducing an equivalence  $\tau_{\leq m} B_i \rightarrow \tau_{\leq m} C_i$  for all  $i \geq i_2$ . We have thus shown that  $\tau_{\leq m}(T_i \otimes_{R_i}$

$\pi_0(R_i)$  is a discrete and flat  $\pi_0(R_i)$ -module for all  $i \geq i_2$ . Observing that there are natural equivalences  $(\tau_{\leq m} T_i) \otimes_{\tau_{\leq m} R_i} \pi_0(R_i) \simeq \tau_{\leq m} B_i$ , by [Lur04, Theorem 2.5.2] we conclude that  $\tau_{\leq m} R_i \rightarrow \tau_{\leq m} T_i$  is flat for all  $i \geq i_2$ .

For (2), it suffices to treat the case where  $l < n$ , the case  $l \geq n$  being trivial. Now, let  $i \geq i_0$ , then if the map  $f_i : X_i \rightarrow S_i$  is a relative  $n$ -stack and  $\Delta_{f_i} : X_i \rightarrow X_i \times_{S_i} X_i$  is a relative  $(l-1)$ -stack, then  $f_i$  is automatically a relative  $l$ -stack. Thus, by repeating this argument to the diagonal maps  $\Delta_{f_i}$  it suffices to treat the case that  $l = 0$ . That is, we have to show that if the map  $f : X \rightarrow S$  is a relative algebraic space, then there is an  $i_1 \geq i_0$  such that  $f_{i_1} : X_{i_1} \rightarrow S_{i_1}$  is a relative algebraic space. Note that  $f_i$  (resp.  $f$ ) is a relative algebraic space if and only if  $\tau_{\leq 0} \Delta_{f_i}$  (resp.  $\tau_{\leq 0} \Delta_f$ ) is an isomorphism of algebraic spaces, which makes this result clear.

For (3) and (4) we note that it suffices to consider the case of a morphism of algebraic stacks, and this was shown in [Ryd09, Prop. B.3]. Finally, (5) follows from the combination of (2), (3), and (4).  $\square$

We can now prove Theorem 5.1.

*Proof of Theorem 5.1.* Write  $X = \varinjlim_{\alpha \in A} X_\alpha$  in the category of algebraic stacks, where  $\{X_\alpha : \alpha \in A\}$  ranges over the finitely presented open substacks of  $X$ . Since the morphism of  $\mathcal{S}$ -valued functors  $\mathcal{H}\mathcal{S}_{U/S}^{\text{der}} \hookrightarrow \mathcal{H}\mathcal{S}_{X/S}^{\text{der}}$  is an open immersion for any open immersion of algebraic stacks  $U \rightarrow X$ , it suffices to treat the case where  $\pi : X \rightarrow S$  is finitely presented with quasi-compact and separated diagonal. Next, we apply [LMB, Prop. 4.18(ii)] to reduce to the case where  $R$  is a finitely generated  $\mathbb{Z}$ -algebra and is thus a G-ring. Hence, it suffices to verify the seven conditions of [Lur04, Thm. 7.1.6] to show that  $\mathcal{H}\mathcal{S}_{X/S}^{\text{der}}$  is a derived 1-stack, almost finitely presented over  $R$ .

- (1) The functor  $\mathcal{H}\mathcal{S}_{X/S}^{\text{der}}$  commutes with filtered colimits when restricted to  $k$ -truncated objects of  $\mathcal{S}\mathcal{C}\mathcal{R}_R$ , for each  $k \geq 0$ .

*Proof.* This follows from Lemma 5.12 and [Lur04, Prop. 5.4.9 and 5.4.10].  $\square$

- (2) The functor  $\mathcal{H}\mathcal{S}_{X/S}^{\text{der}}$  is a sheaf for the étale topology.

*Proof.* Clear.  $\square$

- (3) Let  $B$  be a complete, discrete, local, noetherian  $R$ -algebra,  $\mathfrak{m} \triangleleft B$  the maximal ideal. Then the natural map  $\mathcal{H}\mathcal{S}_{X/S}^{\text{der}}(B) \rightarrow \varprojlim_n \mathcal{H}\mathcal{S}_{X/S}^{\text{der}}(B/\mathfrak{m}^n)$  is an equivalence.

*Proof.* This follows from Theorem 4.5.  $\square$

- (4) The functor  $\mathcal{H}\mathcal{S}_{X/S}^{\text{der}}$  has a cotangent complex.

*Proof.* This follows from Proposition 5.11.  $\square$

- (5),(6) The functor  $\mathcal{H}\mathcal{S}_{X/S}^{\text{der}}$  is infinitesimally cohesive, and nilcomplete.

*Proof.* This follows from Lemma 5.8.  $\square$

- (7) For any discrete commutative  $R$ -algebra  $T$ , the space  $\mathcal{H}\mathcal{S}_{X/S}^{\text{der}}(T)$  is 1-truncated.

*Proof.* Clear.  $\square$

$\square$

Before we get to the proof Theorem 2, we will require an analysis of some spaces of sections. Fix a scheme  $T$ , and let  $s : V' \rightarrow V$  be a representable morphism of algebraic  $T$ -stacks. Define the  $T$ -sheaf  $\underline{\mathrm{Sec}}_T(V'/V)$  to be the sections of the morphism  $s$ . We require a mild strengthening of [Ols06, Prop. 5.10] and [Lie06, Lem. 2.10].

**Proposition 5.13.** *Fix a scheme  $T$  and a proper, flat, and finitely presented morphism of algebraic stacks  $p : Z \rightarrow T$  with finite diagonal. Consider a quasi-finite, separated, finitely presented, and representable morphism  $s : Q \rightarrow Z$ , then the  $T$ -sheaf  $\underline{\mathrm{Sec}}_T(Q/Z)$  is a separated and finitely presented algebraic  $T$ -space.*

*Proof.* There is an inclusion of functors  $\underline{\mathrm{Sec}}_T(Q/Z) \subset \underline{\mathrm{Hilb}}_{Q/T}$ , which is represented by open immersions. Thus, by [Ols05, Thm. 1.5],  $\underline{\mathrm{Sec}}_T(Q/Z)$  is a separated and locally finitely presented algebraic  $T$ -space and it remains to show that  $\underline{\mathrm{Sec}}_T(Q/Z)$  is quasi-compact. We may clearly reduce to the case where  $T$  is a reduced noetherian scheme. Also, as  $T$  is noetherian, it suffices to prove the result over a dense open subset of  $T$ . Observe that by Zariski's Main Theorem [LMB, Thm. 16.5(ii)], there is a finite  $Z$ -morphism  $\bar{Q} \rightarrow Z$  and an open immersion  $Q \hookrightarrow \bar{Q}$ . In particular, we see that there is a natural transformation of  $T$ -sheaves  $\underline{\mathrm{Sec}}_T(Q/Z) \rightarrow \underline{\mathrm{Sec}}_T(\bar{Q}/Z)$  which is represented by open immersions. Hence, we may assume for the remainder that the morphism  $s : Q \rightarrow Z$  is finite.

By [Ryd09, Thm. B], there is a finite and surjective morphism  $p : Z' \rightarrow Z$  from a  $T$ -scheme  $Z'$ . Since  $T$  is noetherian and reduced, by [EGA, IV, 6.9.1] we may assume that  $Z' \rightarrow T$  is flat, and take  $Q' = Q \times_Z Z'$ . We have an induced morphism  $p^* : \underline{\mathrm{Sec}}_T(Q/Z) \rightarrow \underline{\mathrm{Sec}}_T(Q'/Z')$ . An application of [Ols06, Prop. 5.10] shows that  $\underline{\mathrm{Sec}}_T(Q'/Z')$  is of finite type, thus it remains to prove that  $p^*$  is finite type.

Since the morphism  $p$  is affine, the formation of  $p_* \mathcal{O}_{Z'}$  commutes with arbitrary base change on  $T$ . By a minor modification to the arguments of [Ols06, 5.11], we may henceforth assume that  $Z$  is reduced, and thus that the morphism  $p^\# : \mathcal{O}_Z \rightarrow p_* \mathcal{O}_{Z'}$  is injective. By passing to a dense open of  $T$ , we may assume that the cokernel of  $p^\#$  is  $T$ -flat, and so we obtain that the morphism  $(p_W)^\# : \mathcal{O}_{Z_W} \rightarrow (p_W)_* \mathcal{O}_{Z'_W}$  is injective for any  $T$ -scheme  $W$ . Hence, we may apply [Ols06, Lem. 5.13] to conclude that the morphism  $p^* : \underline{\mathrm{Sec}}_T(Q/Z) \rightarrow \underline{\mathrm{Sec}}_T(Q'/Z')$  is of finite type.  $\square$

**Corollary 5.14.** *Fix a scheme  $T$ , and a pair of proper, flat, and finitely presented morphisms of algebraic stacks  $p_i : Z_i \rightarrow T$  for  $i = 1, 2$ , with finite diagonals. Given a locally finitely presented morphism of stacks with quasi-compact and separated diagonal  $\pi : X \rightarrow T$ , as well as quasi-finite and representable morphisms  $s_i : Z_i \rightarrow X$  for  $i = 1, 2$ , then the functor on  $\mathbf{Sch}/T$  given by  $T' \mapsto \mathrm{Hom}_{X_{T'}}((Z_1)_{T'}, (Z_2)_{T'})$  is a separated and finitely presented algebraic  $T$ -space. In particular, the open subfunctor  $\mathrm{Isom}_X(Z_1, Z_2) \subset \mathrm{Hom}_X(Z_1, Z_2)$  parameterizing isomorphisms is a separated and finitely presented algebraic  $T$ -space.*

*Proof.* Note that  $\mathrm{Hom}_X(Z_1, Z_2) = \underline{\mathrm{Sec}}_T((Z_1 \times_X Z_2)/Z_1)$  and since the morphism  $\pi$  has quasi-compact and separated diagonal, it follows that for each  $i = 1, 2$ , the morphisms  $s_i$  are separated. Thus, by Proposition 5.13 the functor  $\mathrm{Hom}_X(Z_1, Z_2)$  is a separated and finitely presented algebraic  $T$ -space.  $\square$

*Proof of Theorem 2.* We note that the construction of  $\underline{\mathrm{HS}}_{X/S}$  is local for the Zariski topology  $S$ , so we may assume that  $S = \mathrm{Spec} R$  for some ring  $R$ . Note that  $\underline{\mathrm{HS}}_{X/S} \simeq \tau_{\leq 0} \mathcal{H}S_{X/S}^{\mathrm{der}}$  and so by Theorem 5.1, it follows that  $\underline{\mathrm{HS}}_{X/S}$  is an algebraic stack, locally of finite presentation over  $S$ . That the diagonal morphism  $\Delta_{\underline{\mathrm{HS}}_{X/S}} : \underline{\mathrm{HS}}_{X/S} \rightarrow \underline{\mathrm{HS}}_{X/S} \times_S \underline{\mathrm{HS}}_{X/S}$  is quasi-compact and separated follows from Corollary 5.14.  $\square$



*Proof of Theorem 3.* For an  $S$ -scheme  $T$ , a  $T$ -morphism  $f_T : Z \times_S T \rightarrow X \times_S T$  is equivalent to the data of its graph morphism  $\Gamma_{f_T} : Z \times_S T \rightarrow (Z \times_S X) \times_S T$ . Since the morphism  $\Delta_{X/S} : X \rightarrow X \times_S X$  is quasi-finite, separated, and representable,  $\Gamma_{f_T}$  is quasi-finite, separated, and representable. Thus, we conclude that we have a morphism of stacks  $\Gamma : \mathrm{HOM}_T(Z, X) \rightarrow \underline{\mathrm{HS}}_{X/S}$  given by  $f_T \mapsto \Gamma_{f_T}$ . It remains to show that  $\Gamma$  is represented by open immersions, for which one may apply [Ols06, Lem. 5.2], noting the easy modifications to the argument to treat the stack case.  $\square$

*Proof of Theorem 4.* Clear from Theorem 3.  $\square$

## APPENDIX A. GLUING DERIVED STACKS

In [Lur04, §5.6] it was shown that if we have closed immersions of affine derived schemes  $Z \hookrightarrow U, V$ , then the pushout  $U \amalg_Z V$  exists in the  $\infty$ -category of derived stacks. We will generalize this using the ideas of [AGV08, Appendix A] and prove the following

**Theorem A.1.** *Consider a diagram of derived  $n$ -stacks  $[X_1 \xleftarrow{t_1} X_3 \xrightarrow{t_2} X_2]$ , where the  $t_i$  are closed immersions. Then, the pushout  $X_1 \amalg_{X_3} X_2$  exists in the  $\infty$ -category of derived  $n$ -stacks. Moreover, the pushout  $X_1 \amalg_{X_3} X_2$  has the following uniformity property: given flat (resp. flat and almost finitely presented, resp. smooth) morphisms  $s_i : X'_i \rightarrow X_i$  for  $i = 1, 2, 3$ , and equivalences  $X'_3 \simeq X'_1 \times_{X_1} X_3 \simeq X'_2 \times_{X_2} X_3$ . Then, the induced map  $X'_1 \amalg_{X'_3} X'_2 \rightarrow X_1 \amalg_{X_3} X_2$  is flat (resp. flat and almost finitely presented, resp. smooth), and in the diagram*

$$\begin{array}{ccccc}
 X'_3 & \longrightarrow & X'_2 & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & X'_1 & \longrightarrow & X'_1 \amalg_{X'_3} X'_2 & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \\
 X_3 & \longrightarrow & X_2 & & \\
 & \searrow & \downarrow & \searrow & \\
 & X_1 & \longrightarrow & X_1 \amalg_{X_3} X_2 & 
 \end{array}$$

*all squares, except possibly the top and bottom, are fibered.*

**Remark A.2.** In the notation of Theorem A.1, in the non-derived setting, all of the squares are known to be fibered.

It will be convenient to introduce some definitions that are similar to those given in §2.1.

**Definition A.3.** Fix a diagram of derived stacks  $[X_1 \xleftarrow{t_1} X_3 \xrightarrow{t_2} X_2]$  where the  $t_i$  are closed immersions. Consider the data of a derived stack  $W$  with maps  $\phi_i^W : X_i \rightarrow W$  for  $i = 1, 2, 3$  which are compatible with the morphisms  $t_1$  and  $t_2$ . In a natural way, one can make an  $\infty$ -category  $\mathrm{PO}_{[t_1, t_2]}$  of this data. We say that the data  $(W, \{\phi_i^W\})$  is a

- (1) **categorical pushout** if it is an initial object for the  $\infty$ -category  $\mathrm{PO}_{[t_1, t_2]}$  and the maps  $\phi_i^W$  are closed immersions;
- (2) **uniformly categorical pushout** if for any flat morphism of derived stacks  $W' \rightarrow W$ , the data  $(W', \{(\phi_i^W)_{W'}\})$  is a categorical pushout;
- (3) **strongly uniformly categorical pushout** if it is a uniformly categorical pushout, and given flat (resp. flat and almost finitely presented, resp. smooth) morphisms of derived stacks  $X'_i \rightarrow X_i$  for  $i = 1, 2, 3$ , and equivalences  $X'_3 \simeq X'_1 \times_{X_1} X_3 \simeq X'_2 \times_{X_2} X_3$ , then if there is a uniformly categorical pushout  $(W', \{\phi_i^{W'}\})$  of the

diagram  $[X'_1 \xleftarrow{t'_1} X'_3 \xrightarrow{t'_2} X'_2]$ , the induced map  $W' \rightarrow W$  is flat (resp. flat and almost finitely presented, resp. smooth), and the natural maps  $X'_i \rightarrow W' \times_W X_i$  are equivalences for  $i = 1, 2, 3$ ;

- (4) **ringed pushout** if the induced map on sheaves of rings

$$\mathcal{O}_W \rightarrow (\phi_1^W)_* \mathcal{O}_{X_1} \times_{(\phi_3^W)_* \mathcal{O}_{X_3}} (\phi_2^W)_* \mathcal{O}_{X_2}$$

is an equivalence;

- (5) **uniformly ringed pushout** if for any flat morphism of derived stacks  $W' \rightarrow W$ , the data  $(W', \{(\phi_i^W)_{W'}\})$  is a ringed pushout.

We have the following easy implications:

$$\begin{aligned} \text{strongly uniformly categorical} &\implies \text{uniformly categorical}; \\ \text{ringed} &\implies \text{uniformly ringed}. \end{aligned}$$

Fix simplicial commutative rings  $A, B, C$  and morphisms  $f : A \rightarrow C$ ,  $g : B \rightarrow C$ , let  $R = A \times_C B$ . In [Lur04, §5.6] was defined a functor:

$$\phi_{A,B,C} : \mathcal{M}_R \rightarrow \mathcal{M}_A \times_{\mathcal{M}_C} \mathcal{M}_B.$$

It is given by sending  $N \in \mathcal{M}_R$  to  $N \otimes_R A \in \mathcal{M}_A$ ,  $N \otimes_R B \in \mathcal{M}_B$ , and the equivalence  $\gamma : N \otimes_R A \otimes_A C \simeq N \otimes_R B \otimes_B C$  in  $\mathcal{M}_C$ . The functor  $\phi_{A,B,C}$  admits a right adjoint  $\psi_{A,B,C}$ , which sends a triple  $(N_A, N_B, \gamma : N_A \otimes_A C \simeq N_B \otimes_B C)$  to the  $R$ -module  $N_A \times_{N_C} N_B$ , where  $N_C := N_A \otimes_A C$ . If the maps on discrete rings  $\pi_0(f)$  and  $\pi_0(g)$  are surjections, then Lurie showed in [Lur04, §5.6] that the derived scheme  $\text{Spec}(A \times_C B)$  is a categorical pushout of the diagram  $[\text{Spec } A \rightarrow \text{Spec } C \leftarrow \text{Spec } B]$ , the functor  $\phi$  is fully faithful, and is an equivalence on *connective* objects. In proving Theorem A.1, we will want to employ these affine results, so the following locality result will be useful.

**Lemma A.4.** *For a diagram of derived stacks  $[X_1 \xleftarrow{t_1} X_3 \xrightarrow{t_2} X_2]$ , fix data  $(W, \{\phi_i^W\}) \in \text{PO}_{[t_1, t_2]}$ . Then  $(W, \{\phi_i^W\})$  is a ringed (resp. uniformly categorical, resp. strongly uniformly categorical) pushout if and only if it remains so after fppf base change of derived stacks on  $W$ .*

*Proof.* The claim for ringed pushouts is obvious from the faithful exactness of the fppf pullback of sheaves. For the categorical statement, we note that the necessity of the condition is clear, thus we focus on the sufficiency. So, consider compatible morphisms of derived stacks  $X_i \rightarrow Y$ , and we wish to construct a unique morphism of derived stacks  $W \rightarrow Y$  which is compatible with these maps. Consider an fppf morphism  $U_0 \rightarrow W$ , and let  $U_k$  denote the  $k$ th fiber product of  $U$  over  $W$ . By assumption, we can choose a derived stack  $U_0$  such that  $U_0$  is the categorical pushout of the pulled back diagram, and similarly for the other  $U_k$ . In particular, we obtain unique maps  $U_k \rightarrow Y$  which are all compatible, and so by fppf descent of morphisms, we construct a unique map  $W \rightarrow Y$  which is compatible with the data.

For the strongly uniformly categorical statement, suppose that we are given flat (resp. flat and almost finitely presented, resp. smooth) morphisms  $X'_i \rightarrow X_i$ , equivalences  $X'_3 \simeq X'_1 \times_{X_1} X_3 \simeq X'_2 \times_{X_2} X_3$ , and the categorical pushout  $W' := X'_1 \amalg_{X'_3} X'_2$  exists, and is uniform. That the morphism  $W' \rightarrow W$  is flat (resp. flat and almost finitely presented, resp. smooth) and that the maps  $X'_i \rightarrow W' \times_W X_i$  are equivalences for  $i = 1, 2$ , can be checked fppf locally on  $W$ . By hypothesis, there is an fppf morphism  $U \rightarrow W$  such that the derived stack  $U$  is a strongly uniformly categorical quotient. In particular the pulled back map  $U' \rightarrow W'$  is flat (resp. flat and almost finitely presented, resp. smooth), the morphisms  $X'_i \times_{W'} U' \rightarrow X_i \times_W U$  are flat (resp. flat and almost finitely presented, smooth), thus

the uniformity of the pushout  $W'$  and the strong uniformity of the pushout  $U$  implies that the map  $U' \rightarrow U$  is flat (resp. flat and almost finitely presented, resp. smooth), as required.  $\square$

We may now recast Theorem A.1 as: given a diagram  $[X \leftarrow Z \rightarrow Y]$  of derived stacks, where the maps are closed immersions, then the diagram has a strongly uniformly categorical pushout. We include two easy results that were omitted from [Lur04, §5.6], but will be of use here. This first result is similar to [Fer03, Thm. 2.2(iii)].

**Lemma A.5.** *Let  $A, B, C$  be simplicial commutative rings and suppose we have maps of simplicial rings  $f : A \rightarrow C, g : B \rightarrow C$ , such that the induced map of discrete rings  $\pi_0(f) : \pi_0(A) \rightarrow \pi_0(C)$  is surjective. Let  $R = A \times_C B$ , then there are induced maps of simplicial rings  $f' : A \times_C B \rightarrow A, g' : A \times_C B \rightarrow B$ .*

- (1) *The natural map of discrete rings  $\pi_0(R) \rightarrow \pi_0(A) \times_{\pi_0(C)} \pi_0(B)$  is surjective with square 0 kernel.*
- (2) *The map of discrete rings  $\pi_0(f') : \pi_0(R) \rightarrow \pi_0(A)$  is surjective.*

*Proof.* Note that (2) follows from (1). Thus, to show (1), we observe that there is an exact sequence

$$R \longrightarrow A \oplus B \longrightarrow C$$

which gives rise to the exact sequence of homotopy groups:

$$\pi_1(C) \longrightarrow \pi_0(R) \longrightarrow \pi_0(A) \oplus \pi_0(B) \longrightarrow \pi_0(C).$$

In particular, we see that  $\pi_0(R)$  surjects onto the kernel of the map  $\pi_0(A) \oplus \pi_0(B) \rightarrow \pi_0(C)$ , which is precisely  $\tilde{R} := \pi_0(A) \times_{\pi_0(C)} \pi_0(B)$ . To check that the ideal  $K = \ker(\pi_0(R) \rightarrow \tilde{R})$  is square 0, we let  $h$  be the induced surjection  $\pi_0(R) \rightarrow \pi_0(C)$ , then  $\ker h \supset K$ . Since  $\pi_0(C)$  acts on  $\pi_1(C)$ , and  $\pi_1(C) \rightarrow K$  is surjective, we conclude that  $(\ker h)K = (0)$ , thus  $K^2 = (0)$ .  $\square$

**Lemma A.6.** *Fix a simplicial commutative ring  $A$ , and let  $M$  be an almost connective  $A$ -module. If the  $\pi_0(A)$ -module  $M \otimes_A \pi_0(A)$  is almost perfect, then  $M$  is almost perfect.*

*Proof.* By [Lur04, Prop. 2.5.7(iv)], we have to show that for all  $n$ , there is a perfect  $A$ -module  $N$  (possibly depending on  $n$ ), and an equivalence  $\tau_{\leq n} N \simeq \tau_{\leq n} M$ . Since  $M$  is almost connective, we may thus reduce to the case that  $M$  is bounded. By using truncations, exact sequences, and shifts, we may further assume that  $M$  is a projective  $A$ -module, with  $\tau_{>0} M = \tau_{<0} M = 0$ . Thus, it remains to show that  $M$  is a retract of a finite free  $A$ -module. By [Lur04, Prop. 2.5.3],  $\pi_0(M)$  is a projective  $\pi_0(A)$ -module. Since  $M$  is projective, hence flat, the hypotheses in the Lemma also imply that the almost perfect  $\pi_0(A)$ -module  $M \otimes_A \pi_0(A)$  is discrete, and is equivalent to  $\pi_0(M)[0]$ . Hence,  $\pi_0(M)$  is a finitely generated projective  $\pi_0(A)$ -module and is consequently a retract of a finite free  $\pi_0(A)$ -module  $\pi_0(A)^{\oplus r}$ . The map  $A^{\oplus r} \rightarrow \pi_0(M)$  lifts to a map  $A^{\oplus r} \rightarrow M$ , which is surjective. Thus, since  $M$  is projective,  $M$  is a retract of a finite free  $A$ -module.  $\square$

**Lemma A.7.** *Fix a flat morphism of derived stacks  $f : X \rightarrow Y$ , and suppose that the morphism of 0-truncated stacks  $\tau_{\leq 0} f : \tau_{\leq 0} X \rightarrow \tau_{\leq 0} Y$  is almost finitely presented, then  $f$  is almost finitely presented.*

*Proof.* The statement is smooth local on the source and target of  $f$ , thus we may assume that  $X = \text{Spec } B, Y = \text{Spec } A$  and the flat morphism  $f : X \rightarrow Y$  is induced by flat morphism of simplicial commutative rings  $u : A \rightarrow B$ . By [Lur04, Prop. 3.2.8], it suffices to show that

the cotangent complex  $L_{B/A}$  is almost perfect. By Lemma A.6, since  $L_{B/A}$  is connective, it suffices to show that the  $\pi_0(B)$ -module  $L_{B/A} \otimes_B \pi_0(B)$  is almost perfect. Since  $u$  is flat, then we have an equivalence  $L_{B/A} \otimes_B \pi_0(B) \simeq L_{\pi_0(B)/\pi_0(A)}$ , and the result now follows since  $\pi_0 u$  is finitely presented.  $\square$

We now have two results for discrete rings, with no claim of originality, and are included for lack of knowledge of a precise reference.

**Lemma A.8.** *Fix a surjection of rings  $A \rightarrow A_0$  and let  $I = \ker(A \rightarrow A_0)$ . Suppose that there is a  $k$  such that  $I^k = 0$ .*

- (1) *Given a map of  $A$ -modules  $u : M \rightarrow N$  such that  $u \otimes_A A_0$  is surjective, then  $u$  is surjective.*
- (2) *For an  $A$ -module  $M$ , if  $M \otimes_A A_0$  is finitely generated, then  $M$  is finitely generated.*
- (3) *Given an  $A$ -algebra  $B$ , let  $B_0 = A_0 \otimes_A B$ .*
  - (a) *If  $B_0$  is a finite type  $A_0$ -algebra,  $B$  is a finite type  $A$ -algebra.*
  - (b) *If  $B$  is a flat  $A$ -algebra and  $B_0$  is a finitely presented  $A_0$ -algebra, then  $B$  is a finitely presented  $A$ -algebra.*

*Proof.* For (1), we have by the right exactness of tensor product that  $(\operatorname{coker} u) \otimes_A A_0 = \operatorname{coker} u_0 = (0)$ . Hence,  $\operatorname{coker} u = I \operatorname{coker} u = I^2 \operatorname{coker} u = \cdots = I^k \operatorname{coker} u = (0)$  and  $u$  is surjective. For (2), we pick a surjection  $v_0 : A_0^{\oplus r} \rightarrow M \otimes_A A_0$ , which lifts to a map  $v : A^{\oplus r} \rightarrow M$ , and apply (1) to conclude that  $v$  is a surjection. For (3a), given a surjection  $f_0 : A_0[X] \rightarrow B_0$ , this lifts to a map  $f : A[X] \rightarrow B$ , and again apply (1). For (3b), by (3a), we know that  $A \rightarrow B$  is finite type. For a surjection  $v : A[X] \rightarrow B$ , since  $B$  is  $A$ -flat, we have  $(\ker v) \otimes_A A_0 = \ker(v \otimes_A A_0)$ . Since  $A_0 \rightarrow B_0$  is finitely presented, and  $v_0 = v \otimes_A A_0$  is a surjection, then  $\ker v_0$  is finitely generated as an  $A_0[X]$ -module—we conclude by applying (2).  $\square$

This result is a non-noetherian version of [Kol08, Lem. 44], in the spirit of [Fer03].

**Lemma A.9.** *Fix a diagram of discrete rings  $[A_1 \xleftarrow{t_1} A_3 \xrightarrow{t_2} A_2]$ , where  $t_1$  is surjective. Given flat (resp. faithfully flat, resp. flat and finitely presented, resp. smooth) morphisms  $g_i : A_i \rightarrow A'_i$  for  $i = 1, 2, 3$ , with isomorphisms  $A'_3 \approx A_3 \otimes_{A_1} A'_1 \approx A_3 \otimes_{A_2} A'_2$ . Then the induced map  $g : A_1 \times_{A_3} A_2 \rightarrow A'_1 \times_{A'_3} A'_2$  is flat (resp. faithfully flat, resp. flat and finitely presented, resp. smooth).*

*Proof.* The flatness of the map  $g$  is precisely [Fer03, Thm. 2.2(iv)]. Let  $R = A_1 \times_{A_3} A_2$  and  $R' = A'_1 \times_{A'_3} A'_2$ . Note that  $R' \otimes_R A_i \approx A'_i$  for all  $i$ , also follows from [Fer03, Thm. 2.2]. For the flat and finitely presented condition on  $R \rightarrow R'$ , write  $R' = \varinjlim_j B^j$ , where the  $B^j$  are finite type  $R$ -subalgebras and set  $B_i^j = A_i \otimes_R B^j$ . Note that  $\varinjlim_j B_i^j = A'_i$  for each  $i$ , and since  $A'_i$  is a finitely presented  $A_i$ -algebra, then there is a  $j_0$  sufficiently large so that the map  $B_i^{j_0} \rightarrow A'_i$  admits a section for all  $i$ . By increasing  $j_0$ , we may assume that these sections are compatible, in the sense that the canonical map  $B_1^{j_0} \times_{B_3^{j_0}} B_2^{j_0} \rightarrow R'$  of  $R$ -algebras admits a section, thus is surjective. By [Fer03, Thm. 2.2(iii)], the canonical map  $B^{j_0} \rightarrow B_1^{j_0} \times_{B_3^{j_0}} B_2^{j_0}$  is surjective and so  $R' = B^{j_0}$ , hence is of finite type over  $R$ .

To see that  $R'$  is finitely presented over  $R$ , we let  $q_R : P_R \rightarrow R'$  be a surjection of  $R$ -algebras, where  $P_R$  is a finitely generated polynomial ring over  $R$ . Let  $L = \ker q$ . Next, take  $P_i = P_R \otimes_R A_i$  and observe that  $P_R = P_1 \times_{P_3} P_2$ . Also, we have the induced map  $q_i : P_i \rightarrow A'_i$ . Since each  $A'_i$  is a flat  $A_i$ -algebra, then  $L_i := \ker(q \otimes_P P_i) = (\ker q) \otimes_P P_i$ . Further,

$L_i$  is a finitely generated  $P_i$ -module, since  $A'_i$  is a finitely presented  $A_i$ -algebra. Thus, by applying [Fer03, Thm. 2.2(iv)], we conclude that  $L$  is a finitely generated  $P_R$ -module, and so  $R'$  is a finitely presented  $R$ -algebra. In the case that the maps  $g_i$  are faithfully flat (resp. smooth), showing that  $g$  is faithfully flat (resp. smooth) may be verified on the fibers of points of  $\text{Spec } R$ . Thus, in both cases, it suffices to observe that the map of topological spaces  $\gamma : \text{Spec } A_1 \amalg \text{Spec } A_3 \rightarrow \text{Spec } R$  is surjective, by [Fer03, Thm. 5.1].  $\square$

We finally arrive at the result that ties all of the previous lemmata together.

**Proposition A.10.** *Fix a diagram of derived stacks  $[X_1 \xleftarrow{t_1} X_3 \xrightarrow{t_2} X_2]$ , where the morphisms  $t_i$  are closed immersions. A ringed pushout  $(W, \{\phi_i^W\})$  is strongly uniformly categorical whenever the morphisms  $\phi_i^W : X_i \rightarrow W$  are closed immersions.*

*Proof.* By Lemma A.4, we reduce to the case where  $W = \text{Spec } R$ , for a simplicial commutative ring  $R$ . Since the maps  $\phi_i^W : X_i \rightarrow W$  are all closed immersions, then for  $i = 1, 2, 3$  we may write  $X_i = \text{Spec } A_i$ , for simplicial commutative rings  $A_i$ . The ringed pushout condition is precisely that the natural map  $R \rightarrow A_1 \times_{A_3} A_2$  is an equivalence. By [Lur04, §5.6], the derived scheme  $W = \text{Spec } R$  is the categorical pushout of the diagram. Now, to show that  $W$  is strongly uniformly categorical, we suppose that we are given flat (resp. flat and almost finitely presented, resp. smooth) morphisms  $X'_i \rightarrow X_i$ , equivalences  $X'_3 \simeq X'_2 \times_{X_2} X_3 \simeq X'_1 \times_{X_1} X_3$ , and there is a uniformly categorical pushout  $W'$  of the diagram  $[X'_1 \leftarrow X'_3 \rightarrow X'_2]$ , then we need to show that the map  $W' \rightarrow W$  is flat (resp. flat and almost finitely presented, resp. smooth), and the natural morphisms  $X'_i \rightarrow W' \times_W X_i$  are equivalences.

By the uniformity of the pushout  $W'$ , we may work smooth locally on  $W'$ , thus it suffices to assume that  $W' = \text{Spec } R'$  and  $X'_i = \text{Spec } A'_i$  for simplicial rings  $R'$  and  $A'_i$ . It suffices to show that the map of rings  $R \rightarrow R'$  is flat (resp. flat and almost finitely presented, resp. smooth), whenever the maps of rings  $A_i \rightarrow A'_i$  are flat (resp. flat and almost finitely presented, smooth), and that the natural maps  $\sigma_i : R' \otimes_R A_i \rightarrow A'_i$  are equivalences for  $i = 1, 2, 3$ . Since  $W'$  is the categorical pushout, it follows from [Lur04, §5.6] that there is an equivalence  $R' \rightarrow A'_1 \times_{A'_3} A'_2 = \psi_{A_1, A_2, A_3}(A'_1, A'_2, \delta)$ . By [Lur04, Prop. 5.6.2], we have an equivalence  $\phi_{A_1, A_2, A_3}(R') \simeq (A'_1, A'_2, \delta)$ . This shows precisely that there are equivalences  $R' \otimes_R A_i \rightarrow A'_i$  for all  $i$ . To show that the map  $R \rightarrow R'$  is flat, by [Lur04, Prop. 2.5.2(3)], it suffices to show that if  $N$  is a discrete  $R$ -module, then  $N \otimes_R R'$  is also discrete. Let  $N_i = N \otimes_R A_i$  and  $N'_i = (N \otimes_R R') \otimes_R A_i$ , then we have exact triangles:

$$N \rightarrow N_1 \oplus N_2 \rightarrow N_3 \quad \text{and} \quad N \otimes_R R' \rightarrow N'_1 \oplus N'_2 \rightarrow N'_3.$$

Since  $N$  is discrete, then for any  $j \geq 2$  we have an isomorphism  $\pi_j(N_1) \oplus \pi_j(N_2) \rightarrow \pi_j(N_3)$  and an injection  $\pi_1(N_1) \oplus \pi_1(N_2) \rightarrow \pi_1(N_3)$ . Since  $A_i \rightarrow A'_i$  is flat, then for any  $j$  we have equivalences  $\pi_j(N_i) \otimes_{A_i} A'_i \simeq \pi_j(N_i \otimes_{A_i} A'_i)$ . Since we have already proved that we have equivalences  $N_i \otimes_{A_i} A'_i \simeq (N \otimes_R A_i) \otimes_{A_i} (A_i \otimes_R R') \simeq N'_i$ , we conclude that the map  $\pi_j(N'_1) \oplus \pi_j(N'_2) \rightarrow \pi_j(N'_3)$  is an isomorphism for  $j \geq 2$  and injective for  $j = 1$ , by the flatness of the maps  $A_i \rightarrow A'_i$ . Hence,  $N \otimes_R R'$  is discrete, as required.

If the maps  $A_i \rightarrow A'_i$  are flat and almost finitely presented (resp. smooth), we have already shown that the map  $g : R \rightarrow R'$  is flat, so we need to check that it is almost finitely presented (resp. smooth). By Lemma A.7, it suffices to check that  $\pi_0(g) : \pi_0(R) \rightarrow \pi_0(R')$  is finitely presented. The flatness of the map  $\pi_0(g)$  immediately implies that we have a

cocartesian diagram of rings:

$$\begin{array}{ccc} \pi_0(R') & \longrightarrow & \pi_0(A'_1) \times_{\pi_0(A'_3)} \pi_0(A'_2) \\ \uparrow & & \uparrow \\ \pi_0(R) & \longrightarrow & \pi_0(A_1) \times_{\pi_0(A_3)} \pi_0(A_2) \end{array}$$

Combining the three lemmata A.5(1), A.8(3b), and A.9, we obtain the claim.  $\square$

We require a generalization of [EGA, IV, 18.1.1] to the derived setting.

**Lemma A.11.** *Fix a closed immersion of derived schemes  $\iota : Z \hookrightarrow X$  and consider a smooth surjection of derived schemes  $p : U \rightarrow Z$ . Then there is an étale covering  $U' \rightarrow U$  and a smooth surjection of derived schemes  $V \rightarrow X$  together with an equivalence  $U' \rightarrow V \times_X Z$ . In the case that  $X$  and  $U$  are disjoint unions of affine derived schemes, then we may take the étale covering  $U' \rightarrow U$  to be a Zariski covering.*

*Proof.* It suffices to prove the second assertion, which is a local problem, so we may assume that  $X = \text{Spec } A$ ,  $Z = \text{Spec } B$ ,  $U = \text{Spec } R$  and we work on the level of simplicial rings. We have an induced smooth morphism of discrete rings  $\pi_0 B \rightarrow \pi_0 R$  and a surjection of discrete rings  $\pi_0 A \rightarrow \pi_0 B$ . By [EGA, IV, 18.1.1] there is a Zariski covering  $\pi_0 R \rightarrow R'_0$ , a faithfully flat smooth map  $\pi_0 A \rightarrow S_0$ , and an isomorphism of rings  $S_0 \otimes_{\pi_0 A} \pi_0 B \rightarrow R'_0$ . Now, since  $\pi_0 R \rightarrow R'_0$  is a Zariski covering, by [Lur04, Thm. 3.4.13], there is a Zariski covering  $R \rightarrow R'$  and an equivalence  $R' \otimes_R \pi_0 R \rightarrow R'_0$ . By flatness, we note that  $R'_0 = \pi_0 R'$ . Also, by [Lur04, Prop. 3.4.11], there is a faithfully flat smooth morphism  $A \rightarrow S$  together with an equivalence  $S \otimes_A \pi_0 A \rightarrow S_0$ , and by flatness we again have that  $\pi_0 S = S_0$ . Using the argument given in the proof of [Lur04, Prop. 3.4.11], there is an equivalence  $S \otimes_A B \rightarrow R'$  lifting the equivalence  $S_0 \otimes_{\pi_0 A} \pi_0 B \rightarrow \pi_0 R'$ .  $\square$

The result that follows is the derived analog of [AGV08, Lem. A.3.2], and their proof works with only minor changes.

**Lemma A.12.** *Consider closed immersions of derived stacks  $Z \hookrightarrow X_1, X_2$ , then there are smooth surjections from derived schemes  $V_i \rightarrow X_i$  and an equivalence  $V_1 \times_{X_1} Z \rightarrow V_2 \times_{X_2} Z$ . The  $V_i$  may be taken to be disjoint unions of affine derived schemes.*

We may now prove Theorem A.1.

*Proof of Theorem A.1.* First, we consider the case where the  $X_i$  are all affine derived schemes. By Proposition A.10, it suffices to construct a ringed pushout  $(W, \{\phi_i^W\})$  of the diagram, such that the maps  $\phi_i^W : X_i \rightarrow W$  are closed immersions. This is immediate from [Lur04, §5.6] and Lemma A.5(2). Note that this implies that we have ringed pushouts for all disjoint unions of affine derived schemes.

In the general case, by Lemma A.12, there are smooth surjections  $U_i^0 \rightarrow X_i$  from derived schemes  $U_i^0$ , which we may assume to be disjoint unions of affine derived schemes. Moreover, there are equivalences  $U_3^0 \simeq U_1^0 \times_{X_1} X_3 \simeq U_2^0 \times_{X_2} X_3$ . To fix notation, for  $k \geq 1$ , let the derived stack  $U_i^k$  denote the  $k$ -fold fiber product of  $U_i^0$  over  $X_i$ .

We now assume that the derived stacks  $X_i$  are  $n$ -stacks, and we will prove that there is a ringed pushout by induction on  $n$ . For  $n = 0$ , the derived  $n$ -stacks  $X_i$  are derived algebraic spaces. Thus, we take the derived scheme  $U^0$  to denote the ringed pushout of the diagram  $[U_1^0 \leftarrow U_3^0 \rightarrow U_2^0]$ , which we have already constructed. For the moment, we assume that the diagonal maps  $\Delta_i : X_i \rightarrow X_i \times X_i$  are affine, and in this case it follows that

the  $U_i^1$  are disjoint unions of affines, and thus by the case already considered, the diagram  $[U_1^1 \leftarrow U_3^1 \rightarrow U_2^1]$  has a ringed pushout  $U^1$ . By Proposition A.10, ringed pushouts are strongly uniformly categorical and so the two induced maps  $U^1 \rightarrow U^0$  are smooth. Since  $U^0$  and  $U^1$  are constructed via categorical pushouts, we conclude that the data  $[U^1 \rightrightarrows U^0]$  defines a smooth groupoid, and we take  $W$  to be the quotient of this groupoid. By Lemma A.4 it is a ringed pushout. We claim that  $W$  is a derived algebraic space. For this, it suffices to show that the algebraic stack  $\tau_{\leq 0}W$  is an algebraic space. By Lemma A.5(1), there are closed immersions  $\tau_{\leq 0}X_i \hookrightarrow \tau_{\leq 0}W$  for  $i = 1$  and  $2$ , with the union of the images covering  $\tau_{\leq 0}W$ . One concludes immediately that  $\tau_{\leq 0}W$  is an algebraic space, since the  $\tau_{\leq 0}X_i$  are algebraic spaces.

In general, the derived schemes  $U_i^1$  given above have affine diagonal, so by what we've proven for the case of derived schemes with affine diagonal, we can conclude that the diagram  $[U_1^1 \leftarrow U_3^1 \rightarrow U_2^1]$  has a ringed pushout  $U^1$ , and we may repeat the arguments just given to construct a ringed pushout  $W$ , which is a derived algebraic space. We now assume that  $n > 0$  and we have proven the result for all derived  $k$ -stacks, for  $k < n$ . Note that since the derived stacks  $X_i$  are  $n$ -stacks, the derived stacks  $U_i^l$  are  $k'$ -stacks where  $k' < n$  for all integers  $l \geq 0$ . Thus there is a ringed pushout  $U^l$  of the diagram  $[U_1^l \leftarrow U_3^l \rightarrow U_2^l]$ . Take  $W$  to be the derived  $n$ -stack associated to the smooth  $n$ -groupoid described by the  $U^l$ , and we conclude that  $W$  is the ringed pushout of the  $X_i$  and the maps  $X_i \rightarrow W$  are all closed immersions.  $\square$

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